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Mean Square Approximation of Solutions of Stochastic Differential Equations Using Legendre's Polynomials*

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We consider approximation for iterated Stratonovich stochastic integrals using multiple Fourier series in terms of Legendre polynomials. We have obtained approximations of stochastic integrals of the multiplicity 1–5. We consider root-mean-square convergence of approximation of iterated stochastic integrals.

Key words: stochastic differential equations, mean square approximation, Fourier series, Legendre's polynomials, numerical methods, simulation.

Introduction

Let $(\Omega, \mathfrak{F}, P)$ be a complete probabilistic space, $\{\mathfrak{F}_t\}_{t \in [0, T]}$ is a non-decreasing right continuous family of the σ-subalgebras $\mathfrak{F}$. Let us consider the system of stochastic differential equations in the integral form

$$x_t = x_0 + \int_0^t a(x_s, u(s), s)\, ds + \int_0^t B(x_s, s)\, dW_s + \int_0^t \int_\Gamma C(x_s, s, y)\, \tilde{\nu}(dy, ds), \tag{1}$$

where $x_s \in \mathbb{R}^n$ is a solution of system (1); $W_s \in \mathbb{R}^m$ is a $\mathfrak{F}_t$-measurable for all $s \in [0, T]$ standard Wiener process with the independent components; $\Gamma = \mathbb{R}^d \setminus \{0\}$;

$$\tilde{\nu}(dy, ds) = \nu(dy, ds) - \pi(dy)\, ds,$$

$\nu(dy, ds)$ is an integral Poisson measure on $\Gamma \times [0, T]$; $M\{\nu(dy, ds)\} = \pi(dy)\, ds$; $\pi(dy)$ is a measure of the intensity $\nu(dy, ds)$, at that $\pi(\Gamma) < \infty$; $a: \mathbb{R}^n \times \mathbb{R}^k \times [0, T] \to \mathbb{R}^n$; $B: \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m}$, $C: \mathbb{R}^n \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^n$ are matrix functions; $x_0 \in \mathbb{R}^n$ is an initial condition; $u: [0, T] \to \mathbb{R}^k$ is a controlling action. The second integral in the right-hand member of (1) is understood in the sense of Ito, and the third one — in the following way [1]:

$$\int_0^t \int_\Gamma C(x_s, s, y)\, \tilde{\nu}(dy, ds) = \lim_{\substack{\Delta_n \to 0 \\ n \to \infty}} \sum_{j=0}^{n-1} \int_\Gamma C(x_{\tau_j}, \tau_j, y)\, \tilde{\nu}(dy, [\tau_j, \tau_{j+1})), \tag{2}$$

where $0 = \tau_0 < \tau_1 < \ldots < \tau_n = t$; $\Delta_n = \max\limits_{0 \le j \le n-1} |\tau_{j+1} - \tau_j|$; the limit is understood in the mean square sense or in probability depending on properties of the random function $C(x_t, t, y)$. Let us also suppose that the functions a, B, C satisfy the condition of existence and uniqueness of solution of system (1) [1].

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It is known [1–4] that when $C \equiv 0$, various particular cases of equations system (1) are adequate mathematical models of a series of physical and technical systems. This system of stochastic differential equations is also used as a mathematical model, particularly, in financial mathematics [2], if the item $\int_0^t \int_\Gamma C(x_s, s, y)\, \tilde{\nu}(dy, ds)$ is present in (1).

One of the effective approaches of numerical integration of system (1) when $C \equiv 0$ is based on the Taylor–Ito or Taylor–Stratonovich expansions which contain iterated integrals of the form:

$$\int_t^s \psi_k(t_k) \dots \int_t^{t_2} \psi_1(t_1)\, dW_{t_1}^{(i_1)} \dots dW_{t_k}^{(i_k)}, \tag{3}$$

$$\int_t^{*s} \psi_k(t_k) \dots \int_t^{*t_2} \psi_1(t_1)\, dW_{t_1}^{(i_1)} \dots dW_{t_k}^{(i_k)}; \tag{4}$$

$\psi_1, \dots, \psi_k$ are continuously differentiable functions on the interval $[t, s]$; $i_1, \dots, i_k = 0, 1, \dots, m$; $W_t^{(i)}$, $i = 1, \dots, m$, are independent standard Wiener processes; $W_t^{(0)} \overset{\text{def}}{=} t$; $\int^*$ is a stochastic Stratonovich integral; $\int$ is a stochastic Ito integral. At that, it is supposed

$$\int_a^{*b} g(s)\, dW_s^{(0)} \overset{\text{def}}{=} \int_a^b g(s)\, ds, \quad \int_a^b g(s)\, dW_s^{(0)} \overset{def}{=} \int_a^b g(s)\, ds.$$

Work [11] proposed the method of numerical integration of system (1) which was based on the Taylor–Ito expansion and Monte Carlo method. The idea of this approach is the following. One divides the interval of integration $[0, T]$ by the points $0 = \tau_0 < \tau_1 < \dots < \tau_n = T$ so that all moments of jumps of the Poisson process $\nu(\Gamma, [0, t))$ would be among τ_j, $j = 0, 1, \dots, n$. If at the point τ_0 the jump $\nu(\Gamma, [0, t))$ is not present, then on the interval $[0, \tau_0]$ the process x_t will be the diffusion one and described by the Ito stochastic differential equation of the form:

$$dx_t = (a(x_t, u(t), t) - \int_\Gamma C(x_t, t, y)\, \pi(dy))\, dt + B(x_t, t)\, dW_t.$$

It can be modeled at the moment τ_0 by means of, for example, the numerical methods based on the Taylor–Ito expansion. If there is a jump of the process x_t at the point τ_0, then $x_{\tau_0}^-$ ($x_t^- \overset{\text{def}}{=} x_{t-0}$) is modeled numerically by means of the numerical methods based on the Taylor–Ito expansion and it is supposed

$$x_{\tau_0} = x_{\tau_0}^- + \int_\Gamma C(x_{\tau_0}^-, \tau_0, y)\, \nu(dy, \{\tau_0\}). \tag{5}$$

The integral in the right-hand of (5) is simulated by means of the Monte-Carlo method. Further, we pass to the time moment τ_1 and follow the described procedures, then we proceed to τ_2 an so on.

In such a way; numerical solving not only Ito stochastic differential equations but also stochastic differential equations in the more general form is reduced at some stage to simultaneous numerical simulation of systems of iterated stochastic integrals of form (3), (4). And at the same time, it is known [5, 6, 14, 15] that every integral (3) can be expressed as an integrals of form (4) of the multiplicity $q \leq k$ and non-stochastic summands.

In [16] the theorem about expanding of an Stratonovich's iterated stochastic integral of form (4) into multiple series expansion by products of standard Gaussian stochastic variables was proved. The theorem gives an effective method of mean-square approximation of this integral.

Let us cite a formulation of the theorem.

Theorem. Let the conditions:

1) $\psi_1(\theta), \ldots, \psi_k(\theta)$ are continuously differentiable functions on the interval $[t, s]$;

2) $\{\varphi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal in the space $L_2([t, s])$ system of continuously differentiable functions. The Fourier series $\sum_{j=0}^{\infty} C_j \varphi_j(x)$, $C_j = \int_t^s f(y)\varphi_j(y)\,dy$, for any piecewise-smooth function $f(x)$ of this system on the open interval (t, s) and bounded on the interval $[t, s]$ converges for any $x \in (t, s)$ to $\frac{1}{2}(f(x+0) + f(x-0))$, has a finite value at $x = t, s$ and converges uniformly to $f(x)$ in any closed interval of continuity of $f(x)$, be fulfilled.

Then

$$J_{s,t}^{*(k)} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{(j_l)\,s,t}^{(i_l)}, \tag{6}$$

where

$$J_{s,t}^{*(k)} = \int_t^{*s} \psi_k(t_k) \ldots \int_t^{*t_2} \psi_1(t_1)\, dW_{t_1}^{(i_1)} \ldots dW_{t_k}^{(i_k)};$$

$\zeta_{(j_l)\,s,t}^{(i_l)} = \int_t^s \varphi_{j_l}(\theta)\, dW_\theta^{(i_l)}$ are independent for different i_l and j_l (when $i_l \neq 0$) standard Gaussian stochastic variables;

$$C_{j_k \ldots j_1} = \int_{[t,s]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \varphi_{j_l}(t_l)\, dt_1 \ldots dt_k;$$

$$K(t_1, \ldots, t_k) = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} 1_{\{t_l < t_{l+1}\}},\ t_1, \ldots, t_k \in [t, s],\ k \geqslant 2.$$

Series (6) converges in the mean-square sense; 1_A is an indicator of the set A; $[t, s]^k \overset{\text{def}}{=} \underbrace{[t, s] \times \ldots \times [t, s]}_{k-1}$.

Work [15] shows that series (6) converges not only in the mean-square sense but also in mean of the degree n; $n \in \mathbb{N}$, $n \geqslant 2$.

The theorem gives an effective method of approximation of the integrals $J_{s,t}^{*(k)}$:

$$J_{s,t}^{*(k)\,q} = \sum_{j_1, \ldots, j_k = 0}^{q} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{(j_l)\,s,t}^{(i_l)}, \tag{7}$$

where $J_{s,t}^{*(k)\,q}$ is an approximation of the integral $J_{s,t}^{*(k)}$, and $q < \infty$ is chosen from the condition

$$M\{(J_{s,t}^{*(k)} - J_{s,t}^{*(k)\,q})^2\} \leqslant \varepsilon;\ \varepsilon \geqslant 0.$$

Work [16] cites a simple formula for computation of $M\{(J_{s,t}^{*(k)} - J_{s,t}^{*(k)\,q})^2\}$ at pairwise different $i_1, \ldots, i_k = 1, \ldots, m$:

$$M\{(J_{s,t}^{*(k)} - J_{s,t}^{*(k)\,q})^2\} = \int_{[t,s]^k} K^2(t_1, \ldots, t_k)\, dt_1 \ldots dt_k - \sum_{j_1, \ldots, j_k = 0}^{q} C_{j_k \ldots j_1}^2. \tag{8}$$

Work [16] notices (by the example of an stochastic integral of the second multiplicity) that the minimal set of independent standard Gaussian stochastic variables which enters in (7) when using a polynomial system of functions is approximately three times as short in comparison with the similar set when a trigonometrical system of functions is used. Besides, work [16] shows that approximation of form (7) which is obtained using the trigonometrical system of functions converges approximately three times as fast in the mean-square sense than approximation based on integral multiple sums.

The polynomial system of functions has one more advantage over the trigonometrical one at approximation of form (6) since the formulas of form (7) for particular stochastic integrals are simpler. Thus, the mean-square approximations of form (7) obtained with the use of the polynomial system of functions are the most effective. In this paper we aim to construct the approximations of form (7) using the system of Legendre's polynomials for particular stochastic Stratonovich integrals of the multiplicity 1–5. With the help of the constructed approximations one can realize the numerical algorithms of integration of stochastic differential equations of orders of accuracy 0.5, 1.0, 1.5, 2.0, 2.5 in the mean-square sense. These algorithms are represented in numerous literature (see, for example, [5–8, 11, 13–15, 18, 19]).

About the minimal sets of iterated stochastic integrals

Let us extract the minimal sets of iterated stochastic integrals which are necessary for approximations of a solution of the Ito stochastic differential equation (equation (1) when $C \equiv 0$) of the order of convergence 0.5, 1.0, 1.5, 2.0 and 2.5 in the mean-square sense. At that the Taylor–Stratonovich series expansion will be used [12, 14, 15].

Let us designate:

$$B_{0\mu_{k-1}\dots\mu_1}^{(i_{k-1}\dots i_1)} g(x,t) = LB_{\mu_{k-1}\dots\mu_1}^{(i_{k-1}\dots i_1)} g(x,t) - \frac{1}{2}\sum_{i_k=1}^{m} G^{(i_k)}G^{(i_k)}B_{\mu_{k-1}\dots\mu_1}^{(i_{k-1}\dots i_1)} g(x,t),$$

$$B_{1\mu_{k-1}\dots\mu_1}^{(i_k\, i_{k-1}\dots i_1)} g(x,t) = G^{(i_k)}B_{\mu_{k-1}\dots\mu_1}^{(i_{k-1}\dots i_1)} g(x,t),\ k = 2,3,\dots;$$

$$B_0 g(x,t) = Lg(x,t) - \frac{1}{2}\sum_{i_1=1}^{m} G^{(i_k)}G^{(i_k)} g(x,t);\ B_1^{(i_1)} g(x,t) = G^{(i_1)} g(x,t);$$

$$L_{x,t}\, g(x,t) = \frac{\partial}{\partial t} g(x,t) + \sum_{i=1}^{n} a^{(i)}(x,t)\frac{\partial}{\partial x^{(i)}}\, g(x,t) + \frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n} B^{(lj)}(x,t)\, B^{(ij)}(x,t)\frac{\partial^2}{\partial x^{(l)}\,\partial x^{(i)}} g(x,t),$$

$$G^{(j)} g(x,t) = \sum_{j=1}^{n} B^{(ji)}(x,t)\frac{\partial}{\partial x^{(j)}} g(x,t);\ i = 1,\dots,m;$$

$${}^{(k)}A \overset{\text{def}}{=} \|A^{(i_1\dots i_k)}\|_{i_1=1,\dots,i_k=1}^{m_1,\dots,m_k};\ m_1,\dots,m_k \geqslant 1;$$

$${}^{(k)}A \overset{k}{\cdot} {}^{(k)}C \overset{\text{def}}{=} \sum_{i_1=1}^{m_1}\dots\sum_{i_k=1}^{m_k} A^{(i_1\dots i_k)}\, C^{(i_1\dots i_k)};$$

$${}^{(p_k)}J^{*}_{(\mu_k\dots\mu_1)\,s,t} = \|J^{*(i_k\dots i_1)}_{(\mu_k\dots\mu_1)\,s,t}\|_{i_1=\delta_1,\dots,i_k=\delta_k}^{m\delta_1,\dots,m\delta_k};$$

$$J^{*(i_k\dots i_1)}_{(\mu_k\dots\mu_1)\,s,t} = \int_{t}^{*s}\dots\int_{t}^{*t_2} dW_{t_1}^{(i_k)}\dots dW_{t_k}^{(i_1)};$$

$\mu_l = 0$ or 1; $\delta_l = 0$ and $i_l = 0$ when $\mu_l = 0$; $\delta_l = 1$ and $i_l = 1, \ldots, m$ when $\mu_l = 1$; $p_k = \sum_{j=1}^{k} \delta_j$; $l = 1, \ldots, k$; $g(x, t)$ is a multiply continuously differentiable function with respect to x and t.

Let us write down the numerical method of the order 2.5 in the mean-square sense for Ito stochastic differential equations (equation (1) when $C \equiv 0$) which is based on a Taylor–Stratonovich expansion [12, 14, 15]:

$$
\begin{aligned}
y_{p+1} = y_p + {}^{(1)}B_1\{y_p\}^{1} \cdot {}^{(1)}J^{*}_{(1)\tau_{p+1},\tau_p} + B_0\{y_p\}\Delta + {}^{(2)}B_{11}\{y_p\}^{2} \cdot {}^{(2)}J^{*}_{(11)\tau_{p+1},\tau_p} + \\
+ {}^{(1)}B_{01}\{y_p\}^{1} \cdot {}^{(1)}J^{*}_{(01)\tau_{p+1},\tau_p} + {}^{(1)}B_{10}\{y_p\}^{1} \cdot {}^{(1)}J^{*}_{(10)\tau_{p+1},\tau_p} + \\
+ {}^{(3)}B_{111}\{y_p\}^{3} \cdot {}^{(3)}J^{*}_{(111)\tau_{p+1},\tau_p} + {}^{(2)}B_{101}\{y_p\}^{2} \cdot {}^{(2)}J^{*}_{(101)\tau_{p+1},\tau_p} + \\
+ {}^{(2)}B_{011}\{y_p\}^{2} \cdot {}^{(2)}J^{*}_{(011)\tau_{p+1},\tau_p} + {}^{(2)}B_{110}\{y_p\}^{2} \cdot {}^{(2)}J^{*}_{(110)\tau_{p+1},\tau_p} + B_{00}\{y_p\}\frac{\Delta^2}{2} + \\
+ {}^{(4)}B_{1111}\{y_p\}^{4} \cdot {}^{(4)}J^{*}_{(1111)\tau_{p+1},\tau_p} + {}^{(1)}B_{100}\{y_p\}^{1} \cdot {}^{(1)}J^{*}_{(100)\tau_{p+1},\tau_p} + \\
+ {}^{(1)}B_{010}\{y_p\}^{1} \cdot {}^{(1)}J^{*}_{(010)\tau_{p+1},\tau_p} + {}^{(1)}B_{001}\{y_p\}^{1} \cdot {}^{(1)}J^{*}_{(001)\tau_{p+1},\tau_p} + \\
+ {}^{(3)}B_{1110}\{y_p\}^{3} \cdot {}^{(3)}J^{*}_{(1110)\tau_{p+1},\tau_p} + {}^{(3)}B_{1101}\{y_p\}^{3} \cdot {}^{(3)}J^{*}_{(1101)\tau_{p+1},\tau_p} + \\
+ {}^{(3)}B_{1011}\{y_p\}^{3} \cdot {}^{(3)}J^{*}_{(1011)\tau_{p+1},\tau_p} + {}^{(3)}B_{0111}\{y_p\}^{3} \cdot {}^{(3)}J^{*}_{(0111)\tau_{p+1},\tau_p} + \\
+ {}^{(5)}B_{11111}\{y_p\}^{5} \cdot {}^{(5)}J^{*}_{(11111)\tau_{p+1},\tau_p},
\end{aligned}
\tag{9}
$$

where $\tau_p = p\Delta$; $p = 0, 1, \ldots, N$; $\Delta = T/N$; $y_{\tau_p} \overset{\text{def}}{=} y_p$; we also suppose that the existence condition of the right-hand part of (9) holds true. If in the right-hand part of (9) the first three summands are taken then we obtain the Euler method which converges strongly with the order $\gamma = 0.5$ at certain conditions [5–7] to a solution of the Ito stochastic differential equation, i.e.

$$M\{|x_T - y_T|\} \leq K\Delta^{\gamma},$$

where the constant K does not depend upon Δ. To realize numerically the Euler method we are to simulate numerically the stochastic integrals $J^{*(i)}_{(1)\tau_{p+1},\tau_p}$, $i = 1, \ldots, m$. For that we shall use the simple relationships

$$J^{*(i)}_{(1)\tau_{p+1},\tau_p} = \sqrt{\Delta}\,\zeta_p^{(i)},$$

where $\zeta_p^{(i)}$ are independent at different p or i standard Gaussian stochastic variables.

If in the right-hand part of (9) the first four summands are taken then we come to the G.N. Milstain method which converges strongly with the order $\gamma = 1.0$ at certain conditions [5–7, 14, 15] to a solution

of the Ito stochastic differential equation. To realize numerically this method we are jointly to simulate numerically the stochastic integrals

$$J^{*(i_1)}_{(1)\tau_{p+1},\tau_p}, J^{*(i_2 i_1)}_{(1\,1)\tau_{p+1},\tau_p}, i_1, i_2 = 1, \ldots, m. \tag{10}$$

Let us take in the right-hand part of (9) the first seven summands. As a result we come to the numerical method which converges strongly with the order $\gamma = 1.5$ at certain conditions [5–7, 14, 15] to a solution of the Ito stochastic differential equation. To realize numerically this method we are to simulate numerically the following set of stochastic iterated integrals:

$$J^{*(i_1)}_{(1)\tau_{p+1},\tau_p}, J^{*(i_2 i_1)}_{(1\,1)\tau_{p+1},\tau_p}, J^{*(0 i_1)}_{(0\,1)\tau_{p+1},\tau_p}, J^{*(i_2 0)}_{(1\,0)\tau_{p+1},\tau_p}, J^{*(i_3 i_2 i_1)}_{(1\,1\,1)\tau_{p+1},\tau_p}, \tag{11}$$

$$i_1, i_2, i_3 = 1, \ldots, m.$$

Set (11) is not minimal since one of the integrals $J^{*(0 i_1)}_{(0\,1)\tau_{p+1},\tau_p}$, $J^{*(i_2 0)}_{(1\,0)\tau_{p+1},\tau_p}$ can be expressed in terms of another. Really, using lemma 2 [16] and the theorem about change of the order of integration in Ito stochastic iterated integrals [14, 15, 20], one can show that

$$J^{*(0 i_1)}_{(0\,1)\tau_{p+1},\tau_p} = -I^{*(i_1)}_{1\,\tau_{p+1},\tau_p} \quad \text{almost certainly (a.c.)},$$

$$J^{*(i_2 0)}_{(1\,0)\tau_{p+1},\tau_p} = \Delta I^{*(i_2)}_{0\,\tau_{p+1},\tau_p} + I^{*(i_2)}_{1\,\tau_{p+1},\tau_p} \quad \text{a.c.}$$

Here and further

$$I^{*(i_k \ldots i_1)}_{l_k \ldots l_1\,s,t} \stackrel{\text{def}}{=} \int\limits_t^{*s} (t - t_k)^{l_1} \ldots \int\limits_t^{*t_2} (t - t_1)^{l_k} dW^{(i_k)}_{t_1} \ldots dW^{(i_1)}_{t_k};$$

$$i_1, \ldots, i_k = 1, \ldots, m; l_k, \ldots, l_1 \in \mathbb{N}.$$

Consequently, the minimal set of stochastic iterated integrals which is necessary for realization of the strong numerical method of the order 1.5 has the form

$$I^{*(i_1)}_{0\,\tau_{p+1},\tau_p}, I^{*(i_2 i_1)}_{00\,\tau_{p+1},\tau_p}, I^{*(i_1)}_{1\,\tau_{p+1},\tau_p}, I^{*(i_3 i_2 i_1)}_{000\,\tau_{p+1},\tau_p}; i_1, i_2, i_3 = 1, \ldots, m. \tag{12}$$

Let us pass to consideration of the minimal sets of stochastic iterated integrals which are necessary for realization of the strong numerical methods of the orders 2.0 and 2.5.

By lemma 2 [16] and the theorem about change of the order of integration in Ito stochastic iterated integrals [14, 15, 20] we get:

$$J^{*(0 i_2 i_1)}_{(0\,1\,1)\tau_{p+1},\tau_p} = -I^{*(i_2 i_1)}_{10\,\tau_{p+1},\tau_p} \quad \text{a.c.}, \tag{13}$$

$$J^{*(i_3 i_2 0)}_{(1\,1\,0)\tau_{p+1},\tau_p} = \Delta I^{*(i_3 i_2)}_{00\,\tau_{p+1},\tau_p} + I^{*(i_3 i_2)}_{01\,\tau_{p+1},\tau_p} \quad \text{a.c.}, \tag{14}$$

$$J^{*(i_3 0 i_1)}_{(1\,0\,1)\tau_{p+1},\tau_p} = I^{*(i_3 i_1)}_{10\,\tau_{p+1},\tau_p} - I^{*(i_3 i_1)}_{01\,\tau_{p+1},\tau_p} \quad \text{a.c.} \tag{15}$$

Let us take in the right-hand part of (9) the first 12 summands. As a result we come to the numerical method which converges strongly with the order $\gamma = 2.0$ at certain conditions [5–7, 14, 15] to a solution of the Ito stochastic differential equation. Therefore, according to (13)–(15) the minimal set of stochastic iterated integrals which is necessary for realization of this numerical method has the following form:

$$I_{0\,\tau_{p+1},\tau_p}^{*(i_1)},\ I_{00\,\tau_{p+1},\tau_p}^{*(i_2 i_1)},\ I_{1\,\tau_{p+1},\tau_p}^{*(i_1)},\ I_{000\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)},\ I_{10\,\tau_{p+1},\tau_p}^{*(i_2 i_1)},$$

$$I_{01\,\tau_{p+1},\tau_p}^{*(i_2 i_1)},\ I_{0000\,\tau_{p+1},\tau_p}^{*(i_4 i_3 i_2 i_1)};\ i_1, i_2, i_3, i_4 = 1, \ldots, m. \tag{16}$$

Developing this approach, we get:

$$J_{(1\,0\,0)\tau_{p+1},\tau_p}^{*(i_3 0 0)} = \frac{\Delta^2}{2} I_{0\,\tau_{p+1},\tau_p}^{*(i_3)} + \Delta I_{1\,\tau_{p+1},\tau_p}^{*(i_3)} + \frac{1}{2} I_{2\,\tau_{p+1},\tau_p}^{*(i_3)} \quad \text{a.c.}, \tag{17}$$

$$J_{(0\,1\,0)\tau_{p+1},\tau_p}^{*(0 i_2 0)} = -\Delta I_{1\,\tau_{p+1},\tau_p}^{*(i_2)} - I_{2\,\tau_{p+1},\tau_p}^{*(i_2)} \quad \text{a.c.}, \tag{18}$$

$$J_{(0\,0\,1)\tau_{p+1},\tau_p}^{*(0 0 i_1)} = \frac{1}{2} I_{2\,\tau_{p+1},\tau_p}^{*(i_1)} \quad \text{a.c.}, \tag{19}$$

$$J_{(1\,1\,1\,0)\tau_{p+1},\tau_p}^{*(i_4 i_3 i_2 0)} = \Delta I_{000\,\tau_{p+1},\tau_p}^{*(i_4 i_3 i_2)} + I_{001\,\tau_{p+1},\tau_p}^{*(i_4 i_3 i_2)} \quad \text{a.c.}, \tag{20}$$

$$J_{(0\,1\,1\,1)\tau_{p+1},\tau_p}^{*(0 i_3 i_2 i_1)} = -I_{100\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)} \quad \text{a.c.}, \tag{21}$$

$$J_{(1\,0\,1\,1)\tau_{p+1},\tau_p}^{*(i_4 0 i_2 i_1)} = -I_{010\,\tau_{p+1},\tau_p}^{*(i_4 i_2 i_1)} + I_{100\,\tau_{p+1},\tau_p}^{*(i_4 i_2 i_1)} \quad \text{a.c.}, \tag{22}$$

$$J_{(1\,1\,0\,1)\tau_{p+1},\tau_p}^{*(i_4 i_3 0 i_1)} = I_{010\,\tau_{p+1},\tau_p}^{*(i_4 i_3 i_1)} + I_{001\,\tau_{p+1},\tau_p}^{*(i_4 i_3 i_1)} \quad \text{a.c.} \tag{23}$$

It follows from (16)–(23) that the minimal set of stochastic iterated integrals which is necessary for realization of numerical method (9) has the following form:

$$I_{0\,\tau_{p+1},\tau_p}^{*(i_1)},\ I_{00\,\tau_{p+1},\tau_p}^{*(i_2 i_1)},\ I_{1\,\tau_{p+1},\tau_p}^{*(i_1)},\ I_{000\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)},\ I_{10\,\tau_{p+1},\tau_p}^{*(i_2 i_1)},\ I_{01\,\tau_{p+1},\tau_p}^{*(i_2 i_1)},$$

$$I_{0000\,\tau_{p+1},\tau_p}^{*(i_4 i_3 i_2 i_1)},\ I_{2\,\tau_{p+1},\tau_p}^{*(i_1)},\ I_{001\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)},\ I_{100\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)},\ I_{010\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)},\ I_{00000\,\tau_{p+1},\tau_p}^{*(i_5 i_4 i_3 i_2 i_1)}. \tag{24}$$

Further sets (10), (12), (16), (24) will be called correspondingly the 1.0-, 1.5-, 2.0-, 2.5-minimal sets of stochastic integrals.

Numerical simulation of the 1.0-minimal set of stochastic integrals

To construct the approximations of form (7) we shall use the complete orthonormal in the space $L_2([\tau_p, \tau_{p+1}])$ system of the Legendre polynomials:

$$\Phi_j(x) = \frac{\sqrt{2j+1}}{\Delta} P_j\left(\left(x - \frac{2\tau_p + \Delta}{2}\right)\frac{2}{\Delta}\right),\ j = 0, 1, 2, \ldots,$$

where $P_j(x)$ is a classical Legendre polynomial:

$$P_j(x) = \frac{1}{2^j j!} \frac{d^j}{dx^j}((x^2-1)^j), j = 1, 2, \ldots,$$

$$P_0(x) = 1.$$

Let us note some known properties of the polynomials $P_j(x)$, which will be used at computation of coefficients of multiple Fourier series:

$$\frac{d}{dx} P_{j+1}(x) - \frac{d}{dx} P_{j-1}(x) = (2j+1) P_j(x), j = 1, 2, \ldots, \tag{25}$$

$$\int_{-1}^{1} x^k P_j(x)\, dx = 0, k = 0, 1, 2, \ldots, j-1, \tag{26}$$

$$\int_{-1}^{1} P_k(x) P_j(x)\, dx = \begin{cases} 0, & k \neq j, \\ \dfrac{2}{2j+1}, & k = j, \end{cases} \tag{27}$$

$$jP_j(x) - (2j-1) xP_{j-1}(x) + (j-1) P_{j-2}(x) = 0, j = 2, 3, \ldots. \tag{28}$$

Using (25)–(28), we obtain the partial form of formula (7) for the stochastic integral $I_{00}^{*(i_2 i_1)}{}_{\tau_{p+1}, \tau_p}$, $i_1 \neq i_2$, $i_1, i_2 = 1, \ldots, m$:

$$I_{00}^{*(i_2 i_1) q}{}_{\tau_{p+1}, \tau_p} = \frac{\Delta}{2} \left[\zeta_0^{(i_1)} \zeta_0^{(i_2)} + \sum_{i=1}^{q} \frac{1}{\sqrt{4i^2-1}} (\zeta_{i-1}^{(i_2)} \zeta_i^{(i_1)} - \zeta_i^{(i_2)} \zeta_{i-1}^{(i_1)}) \right], \tag{29}$$

where $\zeta_j^{(i)} \overset{\text{def}}{=} \int_{\tau_p}^{\tau_{p+1}} \Phi_j(s)\, dW_s^{(i)}$.

If $i_1 = i_2 \neq 0$, then using the Ito formula or (29) when $i_1 = i_2$, $q = \infty$, we obtain

$$I_{00}^{*(i_1 i_1)}{}_{\tau_{p+1}, \tau_p} = \frac{\Delta}{2} (\zeta_0^{(i_1)})^2 \text{ a.c.} \tag{30}$$

Besides, let us set

$$I_0^{*(i_1)}{}_{\tau_{p+1}, \tau_p} = \sqrt{\Delta}\, \zeta_0^{(i_1)}. \tag{31}$$

Formulas (29)–(31) enable to simulate numerically the 1.0-minimal set of stochastic integrals. From (8) and (29) we get

$$M\{(I_{00}^{*(i_2 i_1)}{}_{\tau_{p+1}, \tau_p} - I_{00}^{*(i_2 i_1) q}{}_{\tau_{p+1}, \tau_p})^2\} = \frac{\Delta^2}{2} \left(\frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2-1} \right). \tag{32}$$

The condition [5–7, 14, 15]:

$$M\{(I_{00}^{*(i_2 i_1)}{}_{\tau_{p+1}, \tau_p} - I_{00}^{*(i_2 i_1) q}{}_{\tau_{p+1}, \tau_p})^2\} \leq C\Delta^3, \tag{33}$$

where the constant C does not depend upon Δ and is considered as given, is one of the requirements which ensures strong convergence of the order 1.0 of the Milstain method. Combining (32), (33), we come to the following condition for finding of the minimal number q:

$$\frac{1}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\leqslant C\Delta. \tag{34}$$

Numerical simulation of the 1.5-minimal set of stochastic integrals

To simulate numerically the 1.5-minimal set of stochastic integrals, we are to supplement (29)–(31) with formulas for approximations of stochastic integrals

$$I_{1\,\tau_{p+1},\tau_p}^{*(i_1)},\quad I_{000\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)},\quad i_1,i_2,i_3=1,\ldots,m.$$

We have the following relationships:

$$I_{1\,\tau_{p+1},\tau_p}^{*(i_1)}=-\frac{\Delta^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right)\text{ a.c.}, \tag{35}$$

$$I_{000\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)\,q_1}=\sum_{i,j,k=0}^{q_1}C_{ijk}\,\zeta_k^{(i_1)}\zeta_j^{(i_2)}\zeta_i^{(i_3)}, \tag{36}$$

where

$$C_{ijk}=\int_{\tau_p}^{\tau_{p+1}}\int_{\tau_p}^{z}\int_{\tau_p}^{y}\Phi_k(x)\,dx\,\Phi_j(y)\,dy\,\Phi_i(z)\,dz.$$

It is not difficult to see that

$$C_{ijk}=\frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{8}\,\Delta^{3/2}\,\overline{C}_{ijk},$$

where

$$\overline{C}_{ijk}=\int_{-1}^{1}\int_{-1}^{z}\int_{-1}^{y}P_k(x)\,dx\,P_j(y)\,dy\,P_i(z)\,dz.$$

The formula for expansion of the integral $I_{000\,T,t}^{*(i_3 i_2 i_1)}$ by means of Legendre's polynomials is rather complicated when i_1, i_2, i_3 are arbitrary (see Appendix). The number q_1 in (36) can be not great while the error of root-mean-square approximation which is found when $i_1\neq i_2$, $i_1\neq i_3$, $i_2\neq i_3$ by the formula

$$\Delta^3\left(\frac{1}{6}-\sum_{i,j,k=0}^{q_1}\widetilde{C}_{ijk}^{\,2}\right),\quad \widetilde{C}_{ijk}=C_{ijk}/\Delta^{3/2}, \tag{37}$$

will be quite acceptable.

Let us put $q_1=5$. The exact values of the coefficients $\overline{C}_{ijk}$ when $i,j,k=0,1,\ldots,5$ are given in Tables 1 and 2 (accuracy of computations here and further was checked using the symbolic package of mathematical

Table 1

j	k					
	0	1	2	3	4	5
			$\overline{C}_{0jk}$			
0	4/3	-2/3	2/15	0	0	0
1	0	2/15	-2/15	4/105	0	0
2	-4/15	2/15	2/105	-2/35	2/105	0
3	0	-2/35	2/35	2/315	-2/63	8/693
4	0	0	-8/315	2/63	2/693	-2/99
5	0	0	0	-10/693	2/99	2/1287
			$\overline{C}_{1jk}$			
0	2/3	-4/15	0	2/105	0	0
1	2/15	0	-4/105	0	2/315	0
2	-2/15	8/105	0	-2/105	0	4/1155
3	-2/35	0	8/315	0	-38/3465	0
4	0	-4/315	0	46/3465	0	-64/9009
5	0	0	-4/693	0	74/9009	0
			$\overline{C}_{2jk}$			
0	2/15	0	-4/105	0	2/315	0
1	2/15	-4/105	0	-2/315	0	8/3465
2	2/105	0	0	0	-2/495	0
3	-2/35	8/315	0	-2/3465	0	-116/45045
4	-8/315	0	4/495	0	-2/6435	0
5	0	-4/693	0	38/9009	0	-8/45045

transformations DERIVE). Computing the value of expression (37) when $q_1 = 5$ and taking account of the requirement

$$M\{(I_{000}^{*(i_3 i_2 i_1)}{}_{\tau_{p+1},\tau_p} - I_{000}^{*(i_3 i_2 i_1) q_1}{}_{\tau_{p+1},\tau_p})^2\} \leqslant C\Delta^4,$$

which ensures strong convergence of the order 2.0 of the numerical scheme which includes the considered stochastic integrals [5–7, 14, 15] when $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$ we get

$$0.0230976 \leqslant C\Delta, \tag{38}$$

where C, for example, is equal to 1.

Now condition (34) due to a requirement to ensure strong convergence of the order 2.0 of the numerical scheme is to be replaced by the condition

$$\frac{1}{2}\left(\frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2 - 1}\right) \leqslant C\Delta^2. \tag{39}$$

If $i_1 = i_2 = i_3$ then by the Ito formula we obtain

$$I_{000}^{*(i_1 i_1 i_1)}{}_{\tau_{p+1},\tau_p} = \frac{\Delta^3}{6}(\xi_0^{(i_1)})^3 \text{ a.c.} \tag{40}$$

Table 2

j	k					
	0	1	2	3	4	5
			$\overline{C}_{3jk}$			
0	0	2/105	0	-4/315	0	2/693
1	4/105	0	-2/315	0	-8/3465	0
2	2/35	-2/105	0	4/3465	0	-74/45045
3	2/315	0	-2/3465	0	16/45045	0
4	-2/63	46/3465	0	-32/45045	0	2/9009
5	-10/693	0	38/9009	0	-4/9009	0
			$\overline{C}_{4jk}$			
0	0	0	2/315	0	-4/693	0
1	0	2/315	0	-8/3465	0	-10/9009
2	2/105	0	-2/495	0	4/6435	0
3	2/63	-38/3465	0	16/45045	0	2/9009
4	2/693	0	-2/6435	0	0	0
5	-2/99	74/9009	0	-4/9009	0	-2/153153
			$\overline{C}_{5jk}$			
0	0	0	0	2/693	0	-4/1287
1	0	0	8/3465	0	-10/9009	0
2	0	4/1155	0	-74/45045	0	16/45045
3	8/693	0	-116/45045	0	2/9009	0
4	2/99	-64/9009	0	2/9009	0	4/153153
5	2/1287	0	-8/45045	0	-2/153153	0

If $i_1 = i_2 \neq i_3$, $i_1 = i_3 \neq i_2$, $i_2 = i_3 \neq i_1$ the essential part of summands in formula (36) will be absent due to symmetry. The formulas of type (8) or (37) can be obtained also for $i_1 = i_2 \neq i_3$; $i_1 = i_3 \neq i_2$; $i_2 = i_3 \neq i_1$. It may be shown that in this case the minimal number q_1 will be smaller as compared with $i_1 \neq i_2$, $i_2 \neq i_3$, $i_1 \neq i_3$.

Numerical simulation of the 2.0-minimal set of stochastic integrals

The 2.0-minimal set includes besides the stochastic integrals which form the 1.5-minimal set the following stochastic integrals:

$$I_{10\ \tau_{p+1},\tau_p}^{*(i_2 i_1)}, \quad I_{01\ \tau_{p+1},\tau_p}^{*(i_2 i_1)}, \quad I_{0000\ \tau_{p+1},\tau_p}^{*(i_4 i_3 i_2 i_1)}; \quad i_1, \ldots, i_4 = 1, \ldots, m.$$

Further we have:

$$I_{01\ \tau_{p+1},\tau_p}^{*(i_2 i_1)\, q_2} = -\frac{\Delta^2}{4}\left(\frac{4}{3}\zeta_0^{(i_2)}\zeta_0^{(i_1)} + \frac{2}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)} + \frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)} - \right.$$

$$\left. - \frac{1}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)} + \sum_{i=1}^{q_2}\left\{\frac{1}{\sqrt{(2i+1)(2i+3)}}\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)} - \right.\right. \tag{41}$$

$$-\frac{1}{\sqrt{4i^2-1}}\zeta_i^{(i_2)}\zeta_{i-1}^{(i_1)}-\frac{1}{(2i-1)(2i+3)}\zeta_i^{(i_2)}\zeta_i^{(i_1)}+$$

$$+\frac{1}{\sqrt{(2i+1)(2i+5)(2i+3)}}((i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}-(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)})\Big\}\Big),$$

$$I_{10\,\tau_{p+1},\tau_p}^{*(i_2i_1)\,q_2}=-\frac{\Delta^2}{4}\left(\frac{2}{3}\zeta_0^{(i_2)}\zeta_0^{(i_1)}+\frac{1}{3\sqrt{5}}\zeta_2^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+\right.$$

$$+\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}-\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+\sum_{i=1}^{q_2}\left(-\frac{1}{\sqrt{4i^2-1}}\zeta_i^{(i_2)}\zeta_{i-1}^{(i_1)}+\right. \tag{42}$$

$$+\frac{1}{\sqrt{(2i+1)(2i+3)}}\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}+\frac{1}{\sqrt{(2i+1)(2i+5)(2i+3)}}\times$$

$$\left.\left.\times((i+1)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}-(i+2)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)})+\frac{1}{(2i-1)(2i+3)}\zeta_i^{(i_1)}\zeta_i^{(i_2)}\right)\right).$$

In addition by the Ito formula we obtain

$$I_{01\,\tau_{p+1},\tau_p}^{*(i_1i_1)}+I_{10\,\tau_{p+1},\tau_p}^{*(i_1i_1)}=-\frac{\Delta^2}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right)\zeta_0^{(i_1)}\ \text{a.c.} \tag{43}$$

Now, when $i_1\neq i_2$, we have the following condition:

$$M\{(I_{10\,\tau_{p+1},\tau_p}^{*(i_2i_1)}-I_{10\,\tau_{p+1},\tau_p}^{*(i_2i_1)\,q_2})^2\}=M\{(I_{01\,\tau_{p+1},\tau_p}^{*(i_2i_1)}-I_{01\,\tau_{p+1},\tau_p}^{*(i_2i_1)\,q_2})^2\}=$$

$$=\frac{\Delta^4}{16}\left(\frac{7}{9}-\sum_{i=1}^{q_2}\left(\frac{1}{4i^2-1}+\frac{1}{(2i+1)(2i+3)}+\right.\right.$$

$$\left.\left.+\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}+\frac{1}{(2i-1)^2(2i+3)^2}\right)\right)\leqslant C\Delta^5. \tag{44}$$

The value $C\Delta^5$ in the right-hand part of (44) is conditioned by the requirement to ensure strong convergence of the order 2.0 of the numerical scheme which includes the considered stochastic integrals [5–7, 14, 15].

Let us study the case $i_1=i_2$. In [14, 15] the following statement is proved.

Lemma 1. Let W_t be a standard Wiener process, and $\{\Phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuously differentiable functions in the space $L_2([t,T])$. Then

$$M\{\prod_{i=1}^{2k}\int_t^T\Phi_{r_i}(\tau)\,dW_\tau\}=\begin{cases}(2g+1)!!\ \text{when}\ \{r_1,\ldots,r_{2k}\}=\{q_1,q_1,\ldots,q_k,q_k\},\\0\ \text{otherwise},\end{cases}$$

where g is a number of coincidences of numbers in the collection $\{q_1,\ldots,q_k\}$; $g\geqslant 0$; $r_i=0,1,2,\ldots$; $r_i\neq r_j\ (i\neq j)$; $i,j=1,\ldots,2k$.

By lemma 1 when $i_1=i_2$ we obtain the following formula:

$$M\{(I_{10\,\tau_{p+1},\tau_p}^{*(i_1i_1)}-I_{10\,\tau_{p+1},\tau_p}^{*(i_1i_1)\,q_2})^2\}=M\{(I_{01\,\tau_{p+1},\tau_p}^{*(i_1i_1)}-I_{01\,\tau_{p+1},\tau_p}^{*(i_1i_1)\,q_2})^2\}=$$

$$= \frac{\Delta^4}{16}\left(\sum_{i=q_2+1}^{\infty}\left(\frac{1}{(2i+1)(2i+5)(2i+3)^2}+\frac{2}{(2i-1)^2(2i+3)^2}\right)+\right.$$

$$\left.+\left(\sum_{i=q_2+1}^{\infty}\frac{1}{(2i-1)(2i+3)}\right)^2\right)\leqslant C\Delta^5.$$

Let us consider approximation of the stochastic integral $I_{0000}^{*(i_4 i_3 i_2 i_1)}{}_{\tau_{p+1},\tau_p}$. We have

$$I_{0000}^{*(i_4 i_3 i_2 i_1)\,q_3}{}_{\tau_{p+1},\tau_p} = \sum_{i,j,k,l=0}^{q_3} C_{ijkl}\,\zeta_l^{(i_1)}\,\zeta_k^{(i_2)}\,\zeta_j^{(i_3)}\,\zeta_i^{(i_4)}, \tag{45}$$

where

$$C_{ijkl} = \int_{\tau_p}^{\tau_{p+1}}\int_{\tau_p}^{u}\int_{\tau_p}^{z}\int_{\tau_p}^{y}\Phi_l(x)\,dx\,\Phi_k(y)\,dy\,\Phi_j(z)\,dz\,\Phi_i(u)\,du =$$

$$= \frac{\sqrt{(2i+1)(2j+1)(2k+1)(2l+1)}}{16}\,\Delta^2\,\overline{C}_{ijkl},$$

$$\overline{C}_{ijkl} = \int_{-1}^{1}\int_{-1}^{u}\int_{-1}^{z}\int_{-1}^{y}P_l(x)\,dx\,P_k(y)\,dy\,P_j(z)\,dz\,P_i(u)\,du.$$

Let us choose, for example, $q_3 = 2$. The exact values of the coefficients $\overline{C}_{ijkl}$, $i, j, k, l = 0, 1, 2$ are given in Table 3. For i, j, k, l different in pairs we have the following relationship for a choice of the step of integration Δ when $q_3 = 2$:

$$\Delta^4\left(\frac{1}{24}-\sum_{i,j,k,l=0}^{q_3}\widetilde{C}_{ijkl}^{\,2}\right)\leqslant C\Delta^5;\ \widetilde{C}_{ijkl} = C_{ijkl}/\Delta^2. \tag{46}$$

Table 3

k	l			k	l			k	l		
	0	1	2		0	1	2		0	1	2
	$\overline{C}_{00kl}$				$\overline{C}_{01kl}$				$\overline{C}_{21kl}$		
0	2/3	-2/5	2/15	0	2/15	-2/45	-2/105	0	2/21	-2/45	2/315
1	-2/15	2/15	-2/21	1	2/45	-2/105	2/315	1	2/315	2/315	-2/225
2	-2/15	2/35	2/105	2	-2/35	2/63	-2/315	2	-2/105	2/225	2/1155
	$\overline{C}_{10kl}$				$\overline{C}_{11kl}$				$\overline{C}_{12kl}$		
0	2/5	-2/9	2/35	0	2/15	-2/35	0	0	-2/35	2/45	-2/105
1	-2/45	2/35	-2/45	1	2/105	0	-2/315	1	2/63	-2/105	2/225
2	-2/21	2/45	2/315	2	-4/105	2/105	0	2	2/105	-2/225	-2/3465
	$\overline{C}_{02kl}$				$\overline{C}_{20kl}$				$\overline{C}_{22kl}$		
0	-2/15	2/21	-4/105	0	2/15	-2/35	0	0	2/105	-2/315	0
1	2/35	-4/105	2/105	1	2/105	0	-2/315	1	2/315	0	-2/1155
2	4/105	-2/105	0	2	-4/105	2/105	0	2	0	2/3465	0

Using results from Table 3 while computing the left-hand part of (46), we get

$$0.0236084 \leqslant C\Delta. \tag{47}$$

Let us consider the case $i_1 = i_2 = i_3 = i_4$. By Ito formula we get

$$I_{0000\,\tau_{p+1},\tau_p}^{*(i_1 i_1 i_1 i_1)} = \frac{\Delta^2}{24}(\zeta_0^{(i_1)})^4 \text{ a.c.}$$

If i_1, i_2, i_3, i_4 are such that the requirements

1) $i_1 = i_2 = i_3 = i_4$,

2) $i_l \neq i_k$ $(l \neq k), l, k = 1, 2, 3, 4$,

will be not valid, then some of summands in the right-hand side of (45) will be absent due to symmetry. Besides, in this case the number q_3 can be chosen less than in the case of i_1, i_2, i_3, i_4 different in pairs.

Numerical simulation of the 2.5-minimal set of stochastic integrals

The 2.5-minimal set includes besides the stochastic integrals which form the 2.0-minimal set the following stochastic integrals: $I_{2\,\tau_{p+1},\tau_p}^{*(i_1)}$, $I_{001\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)}$, $I_{010\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)}$, $I_{100\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)}$, $I_{00000\,\tau_{p+1},\tau_p}^{*(i_5 i_4 i_3 i_2 i_1)}$, $i_1, \ldots, i_5 = 1, \ldots, m$.

Further we have:

$$I_{2\,\tau_{p+1},\tau_p}^{*(i_1)} = \frac{\Delta^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right) \text{ a.c.},$$

$$I_{001\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)\,q_4} = \sum_{i,j,k=0}^{q_4} C_{ijk}^{001}\,\zeta_k^{(i_1)}\zeta_j^{(i_2)}\zeta_i^{(i_3)},$$

$$I_{010\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)\,q_5} = \sum_{i,j,k=0}^{q_5} C_{ijk}^{010}\,\zeta_k^{(i_1)}\zeta_j^{(i_2)}\zeta_i^{(i_3)},$$

$$I_{100\,\tau_{p+1},\tau_p}^{*(i_3 i_2 i_1)\,q_6} = \sum_{i,j,k=0}^{q_6} C_{ijk}^{100}\,\zeta_k^{(i_1)}\zeta_j^{(i_2)}\zeta_i^{(i_3)},$$

$$I_{00000\,\tau_{p+1},\tau_p}^{*(i_5 i_4 i_3 i_2 i_1)\,q_7} = \sum_{i,j,k,l,r=0}^{q_7} C_{ijklr}\,\zeta_r^{(i_1)}\zeta_l^{(i_2)}\zeta_k^{(i_3)}\zeta_j^{(i_4)}\zeta_i^{(i_5)},$$

where

$$C_{ijk}^{001} = \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{z} \int_{\tau_p}^{y} \Phi_k(x)\,dx\,\Phi_j(y)\,dy\,(\tau_p - z)\,\Phi_i(z)\,dz,$$

$$C_{ijk}^{010} = \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{z} \int_{\tau_p}^{y} \Phi_k(x)\,dx\,\Phi_j(y)\,(\tau_p - y)\,dy\,\Phi_i(z)\,dz,$$

$$C_{ijk}^{100} = \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{z} \int_{\tau_p}^{y} \Phi_k(x)\,(\tau_p - x)\,dx\,\Phi_j(y)\,dy\,\Phi_i(z)\,dz,$$

$$C_{ijklr} = \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{v} \int_{\tau_p}^{u} \int_{\tau_p}^{z} \int_{\tau_p}^{y} \Phi_r(x)\,dx\,\Phi_l(y)\,dy\,\Phi_k(z)\,dz\,\Phi_j(u)\,du\,\Phi_i(v)\,dv.$$

It is not difficult to see that

$$C_{ijk}^{001} = \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{16}\,\Delta^{5/2}\,\overline{C}_{ijk}^{001},$$

$$C_{ijk}^{010} = \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{16}\,\Delta^{5/2}\,\overline{C}_{ijk}^{010},$$

$$C_{ijk}^{100} = \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{16}\,\Delta^{5/2}\,\overline{C}_{ijk}^{100},$$

$$C_{ijklr} = \frac{\sqrt{(2i+1)(2j+1)(2k+1)(2l+1)(2r+1)}}{32}\,\Delta^{5/2}\,\overline{C}_{ijklr},$$

where

$$\overline{C}_{ijk}^{100} = -\int_{-1}^{1} \int_{-1}^{z} \int_{-1}^{y} P_k(x)\,(x+1)\,dx\,P_j(y)\,dy\,P_i(z)\,dz,$$

$$\overline{C}_{ijk}^{010} = -\int_{-1}^{1} \int_{-1}^{z} \int_{-1}^{y} P_k(x)\,dx\,P_j(y)\,(y+1)\,dy\,P_i(z)\,dz,$$

$$\overline{C}_{ijk}^{001} = -\int_{-1}^{1} \int_{-1}^{z} \int_{-1}^{y} P_k(x)\,dx\,P_j(y)\,dy\,P_i(z)\,(z+1)\,dz,$$

$$\overline{C}_{ijklr} = \int_{-1}^{1} \int_{-1}^{v} \int_{-1}^{u} \int_{-1}^{z} \int_{-1}^{y} P_r(x)\,dx\,P_l(y)\,dy\,P_k(z)\,dz\,P_j(u)\,du\,P_i(v)\,dv.$$

Let us choose $q_4 = q_5 = q_6 = 2$, $q_7 = 1$. The exact values of the coefficients $\overline{C}_{ijk}^{100}$, $\overline{C}_{ijk}^{010}$, $\overline{C}_{ijk}^{001}$, $i, j, k = 0, 1, 2$; $\overline{C}_{ijklr}$, $i, j, k, l, r = 0, 1$ are given in Tables 4, 5.

Table 4

j	k			j	k			j	k		
	0	1	2		0	1	2		0	1	2
	$\overline{C}_{0jk}^{001}$				$\overline{C}_{0jk}^{100}$				$\overline{C}_{0jk}^{010}$		
0	-2	14/15	-2/15	0	-2/3	2/15	2/15	0	-4/3	8/15	0
1	-2/15	-2/15	6/35	1	-2/15	-2/45	2/35	1	-4/15	0	8/105
2	2/5	-22/105	-2/105	2	2/15	-2/35	-4/105	2	4/15	-16/105	0
	$\overline{C}_{1jk}^{001}$				$\overline{C}_{1jk}^{100}$				$\overline{C}_{1jk}^{010}$		
0	-6/5	22/45	-2/105	0	-2/5	2/45	2/21	0	-4/5	4/15	4/105
1	-2/9	-2/105	26/315	1	-2/15	-2/105	4/105	1	-4/15	4/105	4/105
2	22/105	-38/315	-2/315	2	2/35	-2/63	-2/105	2	4/35	-8/105	0
	$\overline{C}_{2jk}^{001}$				$\overline{C}_{2jk}^{100}$				$\overline{C}_{2jk}^{010}$		
0	-2/5	2/21	4/105	0	-2/15	-2/105	4/105	0	-4/15	4/105	4/105
1	-22/105	4/105	2/105	1	-2/21	-2/315	2/105	1	-4/21	4/105	4/315
2	0	-2/105	0	2	-2/105	-2/315	0	2	-4/105	0	0

Table 5

l	r		l	r	
	0	1		0	1
	$\overline{C}_{000lr}$			$\overline{C}_{001lr}$	
0	4/15	-8/45	0	0	4/315
1	-4/45	8/105	1	8/315	-2/105
	$\overline{C}_{010lr}$			$\overline{C}_{100lr}$	
0	4/45	-16/315	0	8/45	-4/35
1	-4/315	4/315	1	-16/315	2/45
	$\overline{C}_{110lr}$			$\overline{C}_{101lr}$	
0	8/105	-2/45	0	4/315	0
1	-4/315	4/315	1	4/315	-8/945
	$\overline{C}_{011lr}$			$\overline{C}_{111lr}$	
0	8/315	-4/315	0	2/105	-8/945
1	0	2/945	1	2/945	0

For i, j, k, l, r different in pairs we have:

$$\Delta^5\left(\frac{1}{60}-\sum_{i,j,k=0}^{2}(\widetilde{C}_{ijk}^{100})^2\right)\leqslant C\Delta^6;\ \widetilde{C}_{ijk}^{100}=C_{ijk}^{100}/\Delta^{5/2}, \tag{48}$$

$$\Delta^5\left(\frac{1}{20}-\sum_{i,j,k=0}^{2}(\widetilde{C}_{ijk}^{010})^2\right)\leqslant C\Delta^6;\ \widetilde{C}_{ijk}^{010}=C_{ijk}^{010}/\Delta^{5/2}, \tag{49}$$

$$\Delta^5\left(\frac{1}{10}-\sum_{i,j,k=0}^{2}(\widetilde{C}_{ijk}^{001})^2\right)\leqslant C\Delta^6;\ \widetilde{C}_{ijk}^{001}=C_{ijk}^{001}/\Delta^{5/2}, \tag{50}$$

$$\Delta^5\left(\frac{1}{120}-\sum_{i,j,k,l,r=0}^{1}(\widetilde{C}_{ijklr})^2\right)\leqslant C\Delta^6;\ \widetilde{C}_{ijklr}=C_{ijklr}/\Delta^{5/2}. \tag{51}$$

The value $C\Delta^6$ in the right-hand part of (48)–(51) is conditioned by the requirement to ensure strong convergence of the order 2.5 of the numerical method which includes the prototype stochastic integrals [5–7, 14, 15]. Relationships (48)–(51) enable to estimate the allowable minimal step of integration Δ when $q_4=q_5=q_6=2$, $q_7=1$. If we choose $q_4, q_5, q_6>2$, $q_7>1$ then, obviously, the allowable minimal step of integration Δ will become less. Using the results cited in Tables 4, 5, we shall rewrite (48)–(51) correspondingly in the form

$$0.00815429\leqslant C\Delta,\ 0.0173903\leqslant C\Delta,$$

$$0.0252801\leqslant C\Delta,\ 0.00759105\leqslant C\Delta.$$

In conclusion we note that while simulation of the γ-minimal ($\gamma=1.0,\ 1.5,\ 2.0,\ 2.5$) set of stochastic integrals the minimal length of a consequence of independent standard Gaussian stochastic variables $\xi_j^{(i)}$ which is necessary for every stochastic integral from the γ-minimal set is to be defined by the demand to ensure strong convergence of the order γ of the considered numerical method [5–7, 14, 15]. This demand consists in that the second moments of approximations errors of all stochastic integrals in the γ-minimal set are not to exceed $C\Delta^{\gamma+1}$ [5–7, 14, 15], where the constant C does not depend upon Δ and is considered as given.

Appendix

The formula for the stochastic integral $I_{000_{T,t}}^{*(i_3 i_2 i_1)}$ $(T > t)$ is the following:

$$I_{000_{T,t}}^{*(i_3 i_2 i_1)} = -\frac{1}{T-t}\left(I_{0_{T,t}}^{*(i_1)} I_{10_{T,t}}^{*(i_2 i_3)} + I_{0_{T,t}}^{*(i_3)} I_{10_{T,t}}^{*(i_2 i_1)}\right) +$$

$$+\frac{1}{2} I_{0_{T,t}}^{*(i_1)} \left(I_{00_{T,t}}^{*(i_3 i_2)} - I_{00_{T,t}}^{*(i_2 i_3)}\right) + \frac{1}{6}(T-t)^{3/2}\,\zeta_0^{(i_1)} \times \tag{52}$$

$$\times\left\{\zeta_0^{(i_3)}\left(\frac{1}{\sqrt{5}}\,\zeta_2^{(i_2)} - \sqrt{3}\,\zeta_1^{(i_2)} - \zeta_0^{(i_2)}\right) + \zeta_1^{(i_3)}\zeta_1^{(i_2)}\right\} + D_{T,t}^{(i_3 i_2 i_1)},$$

where

$$D_{T,t}^{(i_3 i_2 i_1)} = -\frac{(T-t)^{3/2}}{4}\sum_{k=1}^{\infty}\left\{\sum_{\substack{j=0}}^{k+1}\sum_{i=0}^{\infty}_{k+j-i=2p\geq 0,\ p\in Z} L_{ijk}\,K_{j,k+1,\frac{k+j-i}{2}}\,\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} +\right.$$

$$+\sum_{\substack{j=k+2}}^{\infty}\sum_{\substack{i=0\\k+j-i=2p\geq 0,\ p\in Z}}^{\infty} L_{ijk}\,K_{k+1,j,\frac{k+j-i}{2}}\,\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} -$$

$$-\left(\sum_{\substack{j=0}}^{k-1}\sum_{\substack{i=0\\k+j-i=2p\geq 2,\ p\in Z}}^{\infty} L_{ijk}\,K_{j,k-1,\frac{k+j-i}{2}-1}\,\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} +\right.$$

$$+\sum_{\substack{j=k}}^{\infty}\sum_{\substack{i=0\\k+j-i=2p\geq 2,\ p\in Z}}^{\infty} L_{ijk}\,K_{k-1,j,\frac{k+j-i}{2}-1}\,\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} +$$

$$+\sum_{\substack{j=0}}^{k+1}\sum_{\substack{i=0\\k+j-i=2p\geq -2,\ p\in Z}}^{\infty} M_{ijk}\,K_{j,k+1,\frac{k+j-i}{2}+1}\,\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} +$$

$$\left.+\sum_{\substack{j=k+2}}^{\infty}\sum_{\substack{i=0\\k+j-i=2p\geq -2,\ p\in Z}}^{\infty} M_{ijk}\,K_{k+1,j,\frac{k+j-i}{2}+1}\,\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}\right) +$$

$$+\sum_{\substack{j=0}}^{k-1}\sum_{\substack{i=0\\k+j-i=2p\geq 0,\ p\in Z}}^{\infty} M_{ijk}\,K_{j,k-1,\frac{k+j-i}{2}}\,\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} +$$

$$\left.+\sum_{\substack{j=k}}^{\infty}\sum_{\substack{i=0\\k+j-i=2p\geq 0,\ p\in Z}}^{\infty} M_{ijk}\,K_{k-1,j,\frac{k+j-i}{2}}\,\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}\right\},$$

$$L_{ijk} = \left(\frac{2j+1}{(2k+1)(2i+1)}\right)^{1/2}\frac{1}{2i+3},\quad M_{ijk} = \left(\frac{2j+1}{(2k+1)(2i+1)}\right)^{1/2}\frac{1}{2i-1},$$

$$K_{m,n,k} = \frac{a_{m-k}\,a_k\,a_{n-k}}{a_{m+n-k}}\,\frac{2n+2m-4k+1}{2n+2m-2k+1};\quad a_k = \frac{(2k-1)!!}{k!};\quad m \leq n.$$

Equality (52) is true in the mean of the degree n; $n \in \mathbb{N}$, $n \geq 2$.

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