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New Representations of Explicit One-Step Numerical Methods for Jump–Diffusion Stochastic Differential Equations

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Abstract–Numerical integration methods for jump–diffusion stochastic differential equations (SDEs) are considered. The numerical methods are constructed by using a special time discretization adapted to the jumps in a Poisson process, which makes it possible to separate the numerical models of diffusion and jump components in solutions to jump–diffusion SDEs. The proposed numerical methods differ from those described in the literature by the differencing methods applied to the diffusion component of a solution to a jump-diffusion SDE. The proposed methods involve unified Itô–Taylor and Stratonovich–Taylor expansions, strong approximations of multiple Stratonovich integrals (MSIs) based on multiple Fourier series, and new weak approximations of the Itô MSIs.

INTRODUCTION

Jump–diffusion SDEs, which are more general than Itô SDEs, are also important for practical applications. In particular, they are used as mathematical models in stochastic financial mathematics to adequately simulate variations in the dynamics of interest rates, forward interest rates, and bond pricing [1].

One of the first attempts at constructing and implementing numerical methods for jump–diffusion SDEs was made in [2] for a rather special scalar jump–diffusion SDE without diffusion and convective terms, in which the Poisson process was used instead of the Poisson measure. In [3], numerical methods were constructed for a scalar jump–diffusion SDE, but their convergence was not discussed. In [4], Euler's method was applied to jump–diffusion SDEs and was shown to be mean-square convergent. The theory of numerical integration of jump–diffusion SDEs was further developed in [5–8]. The approach to numerical integration proposed in [5] relied on a special time discretization, which made it possible to separate numerically the diffusion and jump components of a solution to an SDE at each integration step. Some strong and weak numerical methods for jump–diffusion SDEs based on this approach were developed and shown to be convergent in [5, 7], respectively.

As noted above, a jump–diffusion SDE can be numerically integrated by separating the diffusion and jump components of its solution at each integration step. This means that the diffusion component can be simulated by applying the numerical methods developed for Itô SDEs in numerous studies (e.g., see [9–14]). The differencing of the diffusion component described in this paper involves the unified Itô–Taylor [12,15] and Stratonovich–Taylor [16] expansions. These expansions were obtained in [12, 15, 16] by changing the order of integration in the Itô MSIs [12] and using relations between the Stratonovich and Itô MSIs [10, 12]. The main difference between the unified Itô–Taylor and Stratonovich–Taylor expansions and the Itô–Taylor [10, 17] and Stratonovich–Taylor [10, 18] expansions is that the number of distinct Itô and Stratonovich MSIs is substantially reduced in the respective unified expansions. It is well known that the differencing of the MSIs is the key problem encountered in implementing the numerical methods. In this paper, it is proposed to deal with this problem by applying an efficient strong approximation of the Stratonovich MSIs, based on multiple Fourier series [12, 19, 20], and new weak approximations of the Itô MSIs.

2. STRONG NUMERICAL METHODS FOR JUMP–DIFFUSION SDEs

Define a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a flow of σ-algebras $\{\mathcal{F}_t, t \in [0, T]\}$, where $\mathcal{F}_f \subset \mathcal{F}$. Consider a set of jump–diffusion SDEs:

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{a}(\mathbf{x}_s, s)ds + \int_0^t B(\mathbf{x}_s, s)d\mathbf{f}_s + \int_0^t\int_X \mathbf{c}(\mathbf{x}_s, s, \mathbf{y})\tilde{\nu}(ds, d\mathbf{y}), \tag{2.1}$$

where $\mathbf{x}_s \in \mathbb{R}^n$ is a solution to (2.1), $\mathbf{f}_s$ is an m-dimensional standard Wiener process with independent components $\mathbf{f}_s^{(i)}$ $(i = 1, 2, \ldots, m)$ that is $\mathcal{F}_s$-measurable at every $s \in [0, T]$, and $\mathbb{R}^d\backslash\{0\} \stackrel{\text{def}}{=} X$. Define $\nu(ds, d\mathbf{y})$ as the Poisson measure in $[0, T] \times X$; $\mathbf{M}\{\nu(ds, d\mathbf{y})\} = \Pi(d\mathbf{y})ds$; $\tilde{\nu}(ds, d\mathbf{y}) = \nu(ds, d\mathbf{y}) - \Pi(d\mathbf{y})ds$ as a martingale measure; $\Pi(d\mathbf{y})ds$ as an intensity measure $(\Pi(X) < \infty)$; $\mathbf{a}(\mathbf{x}, s) : \mathbb{R}^n \times [0, T] \longrightarrow \mathbb{R}^n$, $B(\mathbf{x}, s) : \mathbb{R}^n \times [0, T] \longrightarrow \mathbb{R}^{n\times m}$, and $\mathbf{c}(\mathbf{x}, s, \mathbf{y}) : \mathbb{R}^n \times [0, T] \times X \longrightarrow \mathbb{R}^n$ as matrix-valued functions satisfying the existence and uniqueness conditions for the solution to Eq. (2.1) (see [4]); $\tilde{\nu}([0, s], d\mathbf{y})$ as a process that is $\mathcal{F}_s$-measurable at every $s \in [0, T]$ and independent of $\mathbf{f}_s$; and $\mathbf{x}_0 \in \mathbb{R}^n$ as an $\mathcal{F}_0$-measurable random variable for which $\mathbf{M}\{|\mathbf{x}_0|^2\} < \infty$, where $\mathbf{M}$ denotes an expected value. The second integral on the right-hand side of (2.1) is interpreted as an Itô integral; the last one, as a stochastic integral over the martingale measure [4].

The analogue of the Itô–Taylor expansion of the solution to Eq. (2.1) obtained in [5] contains MSIs not only over the Wiener process, but also over the martingale measure. To circumvent an approximation of the MSIs over the martingale measure in constructing numerical methods for Eq. (2.1), a special discretization of $[0, T]$ adapted to the jumps in $\mathbf{x}_t$ $(t \in [0, T])$ was proposed in [5, 7], which made it possible to separate the simulated jump and diffusion components at each integration step. This approach is formulated as follows.

The Poisson process $\nu([0, t), X)$ $(t \in [0, T])$ generates a corresponding sequence of jump times, which obviously are the jump times of the process $\mathbf{x}_t$ $(t \in [0, T])$ as well.

Define the sequence $\{\tau_j\}_{j=0}^{\infty}$ of stopping times such that

$$0 = \tau_0 < \ldots < \tau_{n_T+1} = T, \quad n_T < \infty \text{ w. p. } 1, \tag{2.2}$$

$$\max_{1 \le i \le n_T+1} |\tau_i - \tau_{i-1}| \le \Delta \text{ w. p. } 1, \tag{2.3}$$

where $n_t = \max\{i = 0, 1, \ldots : \tau_i \le t\}$ for any $t \in [0, T]$. (Henceforth, *w. p.* 1 means *with probability* 1.) Assume that the set of $\tau_0, \ldots, \tau_{n_T+1}$ contains every jump time of $\nu([0, t), X)$ $(t \in [0, T])$ that does not exceed T. If the time τ_{i+1} is not a jump time of $\nu([0, t), X)$ $(t \in [0, T])$, it is supposed to be $\mathcal{F}_{\tau_i}$-measurable.

It is obvious that

$$\mathbf{x}_t = \mathbf{x}_{\tau_i} + \int_{\tau_i}^t \tilde{\mathbf{a}}(\mathbf{x}_s, s)ds + \int_{\tau_i}^t B(\mathbf{x}_s, s)d\mathbf{f}_s \text{ w. p. } 1 \tag{2.4}$$

for $t \in [\tau_i, \tau_{i+1})$, where

$$\tilde{\mathbf{a}}(\mathbf{x}, s) = \mathbf{a}(\mathbf{x}, s) - \int_X \mathbf{c}(\mathbf{x}, s, \mathbf{y})\Pi(d\mathbf{y}), \quad i = 0, 1, \ldots, n_T,$$

while

$$\mathbf{x}_{\tau_i} = \mathbf{x}_{\tau_i-} + \int_{\tau_i-}^{\tau_{i+1}-}\int_X \mathbf{c}(\mathbf{x}_{\tau_i-}, \tau_i, \mathbf{y})\nu(ds, d\mathbf{y}) \text{ w. p. } 1, \quad i = 1, 2, \ldots, n_T. \tag{2.5}$$

According to (2.4), the Itô process $\nu([0, t), X)$ ($t \in [0, T]$) is described by the Itô stochastic differential equation between the jump times of $\mathbf{x}_t$ ($t \in [0, T]$). It follows from (2.4) and (2.5) that

$$\mathbf{x}_{\tau_{i+1}-} = \mathbf{x}_{\tau_i} + \int_{\tau_i}^{\tau_{i+1}-} \tilde{\mathbf{a}}(\mathbf{x}_s, s)ds + \int_{\tau_i}^{\tau_{i+1}-} B(\mathbf{x}_s, s)d\mathbf{f}_s,$$
$$\mathbf{x}_{\tau_{i+1}} = \mathbf{x}_{\tau_{i+1}-} + \int_X \mathbf{c}(\mathbf{x}_{\tau_{i+1}-}, \tau_{i+1}, \mathbf{y})\nu(\{\tau_{i+1}\}, d\mathbf{y}) \tag{2.6}$$

w. p. 1, where $i = 0, 1, \ldots, n_T$. According to [3, 7], if τ_{i+1} is the jump time of a Poisson process, the integral in the second equation in (2.6) should be numerically approximated by

$$\int_X \mathbf{c}(\mathbf{x}_{\tau_{i+1}-}, \tau_{i+1}, \mathbf{y})\nu(\{\tau_{i+1}\}, d\mathbf{y}) = \mathbf{c}(\mathbf{x}_{\tau_{i+1}-}, \tau_{i+1}, \xi_i),$$

where ξ_i are independent $\Pi(d\mathbf{y})/\Pi(X)$-distributed random variables. (Recall that the time intervals between consecutive jumps of a Poisson process are independent random variables that are exponentially distributed with parameter $\Pi(X)$.)

The approximation of $\mathbf{x}_{\tau_{i+1}-}$ in (2.6) constructed in [5, 7] relies on the Itô–Taylor expansion [17]. The number of distinct Itô and Stratonovich MSIs in the unified Itô–Taylor and Stratonovich–Taylor expansions obtained in [12, 15, 16] is reduced as compared to that in the respective Itô–Taylor [17] and Stratonovich–Taylor [18] expansions. Since approximation of MSIs is a difficult problem [9, 10, 12, 19–22], the computational costs of numerical methods can be reduced by using the unified Itô–Taylor and Stratonovich–Taylor expansions. However, the unified Itô–Taylor and Stratonovich–Taylor expansions cannot be directly used to approximate $\mathbf{x}_{\tau_{i+1}-}$ in (2.6), because they were obtained for constant points in time [12, 16], rather than for stopping times. Nevertheless, since the sequence of jumps in a Poisson process can be simulated in advance, the sequence of jumps in the process $\mathbf{x}_t$ ($t \in [0, T]$) can be treated as known and constant without loss of generality. Consider the superposition of a predetermined discretization of rank Δ and a precalculated sequence of jump times of $\mathbf{x}_t$ ($t \in [0, T]$) (see [5, 7]) as a discretization $\{\tau_j\}_{j=1}^{n_T+1}$. Then, it is obvious that unified Itô–Taylor and Stratonovich–Taylor expansions can be used to approximate $\mathbf{x}_{\tau_{i+1}-}$ in (2.6) as follows.

Denote by $\mathcal{L}$ the set of functions $R(\mathbf{x}, t) : \mathbb{R}^n \times [0, T] \longrightarrow \mathbb{R}^n$ that are twice differentiable with respect to x and differentiable with respect to t. Define

$$^{(k)}A = \left\|A^{(i_1 \ldots i_k)}\right\|_{i_1=1, \ldots, i_k=1}^{m_1 \quad m_k}, \quad m_1, \ldots, m_k \geq 1,$$

$$^{(k)}A \overset{k}{\cdot} {}^{(k)}B = \sum_{i_1=1}^{m_1} \ldots \sum_{i_k=1}^{m_k} A^{(i_1 \ldots i_k)} B^{(i_1 \ldots i_k)}, \quad k \geq 1.$$

Henceforth, $\overset{0}{\cdot}$ is the sign of conventional multiplication, and $^{(0)}A$ is interpreted as a scalar A. Define

$$\left\|A_{k+1}D_k^{(i_k)}A_k \ldots A_2 D_1^{(i_1)}A_1 R(\mathbf{x}, t)\right\|_{i_1=1, \ldots, i_k=1}^{m \quad m} \overset{\text{def}}{=} {}^{(k)}A_{k+1}D_k A_k \ldots A_2 D_1 A_1\{R(\mathbf{x}, t)\}, \tag{2.7}$$

$$\underbrace{C \ldots C}_{j} R(\mathbf{x}, t) \overset{\text{def}}{=} \begin{cases} C^j R(\mathbf{x}, t), & j \geq 1, \\ R(\mathbf{x}, t), & j = 0, \end{cases} \tag{2.8}$$

where C, A_p, and $D_q^{(i)}$ ($p = 1, 2, \ldots, k+1$, $q = 1, 2, \ldots, k$; $i = 1, 2, \ldots, m$) are operators defined on $\mathcal{L}$. The right-hand sides in (2.7) and (2.8) are supposed to exist.

Define the following operators on $\mathcal{L}$:

$$LR(\mathbf{x}, t) = \frac{\partial R}{\partial t}(\mathbf{x}, t) + \sum_{k=1}^{n} \tilde{\mathbf{a}}^{(k)}(\mathbf{x}, t)\frac{\partial R}{\partial \mathbf{x}^{(k)}}(\mathbf{x}, t) + \frac{1}{2}\sum_{j,k=1}^{n}\sum_{i=1}^{m} B^{(ki)}(\mathbf{x}, t)B^{(ji)}(\mathbf{x}, t)\frac{\partial^2 R}{\partial \mathbf{x}^{(j)}\mathbf{x}^{k}}(\mathbf{x}, t), \tag{2.9}$$

$$G_0^{(i)}R(\mathbf{x}, t) = \sum_{j=1}^{n} B^{(ji)}(\mathbf{x}, t)\frac{\partial R}{\partial \mathbf{x}^{(j)}}(\mathbf{x}, t), \quad i = 1, 2, \ldots, m. \tag{2.10}$$

Note that $L\mathbf{x} = \tilde{\mathbf{a}}(\mathbf{x}, t)$ and $G_0^{(i)}\mathbf{x} = B_i(\mathbf{x}, t)$, where $\mathbf{x} \in \mathbb{R}^n$ and $B_i(\mathbf{x}, t)$ is the ith column of the matrix $B(\mathbf{x}, t)$.

Consider the Itô MSIs

$$I^{(i_1 \ldots i_k)}_{(l_1 \ldots l_k)s,t} = \begin{cases} \int_t^s (t-\tau_k)^{l_k} \ldots \int_t^{\tau_2} (t-\tau_1)^{l_1} d\mathbf{f}^{(i_1)}_{\tau_1} \ldots d\mathbf{f}^{(i_k)}_{\tau_k}, & k \geq 1 \\ 1, \quad k = 0, \end{cases} \tag{2.11}$$

$$J^{(i_1 \ldots i_k)}_{(l_1 \ldots l_k)s,t} = \begin{cases} \int_t^s (s-\tau_k)^{l_k} \ldots \int_t^{\tau_2} (s-\tau_1)^{l_1} d\mathbf{f}^{(i_1)}_{\tau_1} \ldots d\mathbf{f}^{(i_k)}_{\tau_k}, & k \geq 1 \\ 1, \quad k = 0, \end{cases} \tag{2.12}$$

where $i_1, \ldots, i_k = 1, 2, \ldots, m$ and $l_1, \ldots, l_k = 0, 1, \ldots$, and define

$${}^{(k)}I_{(l_1 \ldots l_k)s,t} = \left\| I^{(i_1 \ldots i_k)}_{(l_1 \ldots l_k)s,t} \right\|^{m}_{i_1 = 1, \ldots, i_k = 1}{}^{m}, \quad {}^{(k)}J_{(l_1 \ldots l_k)s,t} = \left\| J^{(i_1 \ldots i_k)}_{(l_1 \ldots l_k)s,t} \right\|^{m}_{i_1 = 1, \ldots, i_k = 1}{}^{m}.$$

A strong approximation of $\mathbf{x}_{\tau_{i+1}-}$ in (2.6) is constructed by using the unified Itô–Taylor expansions [12] in terms of the Itô MSIs defined by (2.11) and (2.12). As a result, the following numerical schemes are obtained:

$$\begin{cases} \mathbf{y}_{\tau_{p+1}-} = \mathbf{y}_{\tau_p} + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k) \in \mathfrak{D}_q} \frac{(\Delta_p)^j}{j!} {}^{(k)}G_{l_1} \ldots G_{l_k} L^j \{\mathbf{y}_{\tau_p}\} \overset{k}{\cdot} {}^{(k)}\hat{I}_{(l_1 \ldots l_k)\tau_{p+1},\tau_p} \\ \mathbf{y}_{\tau_{p+1}} = \mathbf{y}_{\tau_{p+1}-} + \int_X \mathbf{c}(\mathbf{y}_{\tau_{p+1}-}, \tau_{p+1}, \mathbf{y})\nu(\{\tau_{p+1}\}, d\mathbf{y}), \end{cases} \tag{2.13}$$

$$\begin{cases} \mathbf{y}_{\tau_{p+1}-} = \mathbf{y}_{\tau_p} + \sum_{q=1}^{r} \sum_{(k,j,l_1,\ldots,l_k) \in \mathfrak{D}_q} \frac{(\Delta_p)^j}{j!} {}^{(k)}L^j G_{l_1} \ldots G_{l_k} \{\mathbf{y}_{\tau_p}\} \overset{k}{\cdot} {}^{(k)}\hat{J}_{(l_1 \ldots l_k)\tau_{p+1},\tau_p} \\ \mathbf{y}_{\tau_{p+1}} = \mathbf{y}_{\tau_{p+1}-} + \int_X \mathbf{c}(\mathbf{y}_{\tau_{p+1}-}, \tau_{p+1}, \mathbf{y})\nu(\{\tau_{p+1}\}, d\mathbf{y}), \end{cases} \tag{2.14}$$

where $p = 0, 1, \ldots, n_T$, $\tau_{p+1} - \tau_p \overset{\text{def}}{=} \Delta_p$, $G_q^{(i)} = (G_{q-1}^{(i)}L - LG_{q-1}^{(i)})/q$ with $q = 1, 2, \ldots$ and $i = 1, 2, \ldots, m$; $\mathfrak{D}_q = \{(k, j, l_1, \ldots, l_k) : k + 2(j + l_1 + \ldots + l_k) = q, k, j, l_1, \ldots, l_k = 0, 1, \ldots\}$; and the circumflex henceforth denotes an approximation of the corresponding MSI. The partial derivatives and MSIs on the right-hand sides of (2.13) and (2.14) are supposed to exist.

The analogues of (2.13) and (2.14) based on the unified Stratonovich–Taylor expansions described in [16] are constructed as follows.

Define the operator $\bar{L}$ on $\mathcal{L}$ as follows:

$$\bar{L}R(\mathbf{x}, t) = LR(\mathbf{x}, t) - \frac{1}{2}\sum_{j=1}^{m} G_0^{(j)}G_0^{(j)}R(\mathbf{x}, t).$$

Consider the Stratonovich MSIs

$$I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t} = \begin{cases} \int\limits_t^{*s} (t-\tau_k)^{l_k} \ldots \int\limits_t^{*\tau_2} (t-\tau_1)^{l_1} d\mathbf{f}^{(i_1)}_{\tau_1} \ldots d\mathbf{f}^{(i_k)}_{\tau_k}, & k \ge 1, \\ 1, \quad k = 0, \end{cases} \tag{2.15}$$

$$J^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t} = \begin{cases} \int\limits_t^{*s} (s-\tau_k)^{l_k} \ldots \int\limits_t^{*\tau_2} (s-\tau_1)^{l_1} d\mathbf{f}^{(i_1)}_{\tau_1} \ldots d\mathbf{f}^{(i_k)}_{\tau_k}, & k \ge 1, \\ 1, \quad k = 0, \end{cases} \tag{2.16}$$

where $i_1, \ldots, i_k = 1, 2, \ldots, m$, $l_1, \ldots, l_k = 0, 1, \ldots$ and $\int\limits^{*}$ is a Stratonovich stochastic integral. Define

$$^{(k)}I^{*}_{(l_1\ldots l_k)s,t} = \left\| I^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t} \right\|^{m}_{i_1 = 1, \ldots,}{}^{m}_{i_k = 1}, \quad ^{(k)}J^{*}_{(l_1\ldots l_k)s,t} = \left\| J^{*(i_1\ldots i_k)}_{(l_1\ldots l_k)s,t} \right\|^{m}_{i_1 = 1, \ldots,}{}^{m}_{i_k = 1}.$$

The numerical schemes constructed for (2.1) by using the unified Stratonovich–Taylor expansions [12] in terms of the MSIs defined by (2.15) and (2.16) are as follows:

$$\begin{cases} \mathbf{y}_{\tau_{p+1}-} = \mathbf{y}_{\tau_p} + \sum\limits_{q=1}^{r} \sum\limits_{(k, j, l_1, \ldots, l_k) \in \mathfrak{D}_q} \dfrac{(\Delta_p)^j}{j!} {}^{(k)}\bar{G}_{l_1} \ldots \bar{G}_{l_k} \bar{L}^j \{\mathbf{y}_{\tau_p}\} \overset{k}{\cdot} {}^{(k)}\hat{I}^{*}_{(l_1\ldots l_k)\tau_{p+1}, \tau_p}, \\ \mathbf{y}_{\tau_{p+1}} = \mathbf{y}_{\tau_{p+1}-} + \int\limits_X \mathbf{c}(\mathbf{y}_{\tau_{p+1}-}, \tau_{p+1}, \mathbf{y}) \nu(\{\tau_{p+1}\}, d\mathbf{y}), \end{cases} \tag{2.17}$$

$$\begin{cases} \mathbf{y}_{\tau_{p+1}-} = \mathbf{y}_{\tau_p} + \sum\limits_{q=1}^{r} \sum\limits_{(k, j, l_1, \ldots, l_k) \in \mathfrak{D}_q} \dfrac{(\Delta_p)^j}{j!} {}^{(k)}\bar{L}^j \bar{G}_{l_1} \ldots \bar{G}_{l_k} \{\mathbf{y}_{\tau_p}\} \overset{k}{\cdot} {}^{(k)}\hat{J}^{*}_{(l_1\ldots l_k)\tau_{p+1}, \tau_p}, \\ \mathbf{y}_{\tau_{p+1}} = \mathbf{y}_{\tau_{p+1}-} + \int\limits_X \mathbf{c}(\mathbf{y}_{\tau_{p+1}-}, \tau_{p+1}, \mathbf{y}) \nu(\{\tau_{p+1}\}, d\mathbf{y}), \end{cases} \tag{2.18}$$

where $p = 0, 1, \ldots, n_T$, $\tau_{p+1} - \tau_p \overset{\text{def}}{=} \Delta_p$, $\bar{G}^{(i)}_q = (\bar{G}^{(i)}_{q-1}\bar{L} - \bar{L}\bar{G}^{(i)}_{q-1})/q$, where $q = 1, 2, \ldots$ and $i = 1, 2, \ldots, m$ and $\bar{G}^{(i)}_0 \overset{\text{def}}{=} G^{(i)}_0$, where $i = 1, 2, \ldots, m$. The partial derivatives and MSIs on the right-hand sides of (2.17) and (2.18) are supposed to exist.

Note that (2.13), (2.14), (2.17), and (2.18) contain 12, 20, and 33 different MSIs when $r = 5$, 6, and 7, respectively, whereas their analogues in [5] based on the Itô–Taylor [17] expansion contain 17, 29, and 51 different MSIs, respectively. It was already mentioned here that the differencing of MSIs is a difficult problem. Therefore, schemes (2.13), (2.14), (2.17), and (2.18) have a certain advantage over their analogues constructed in [5].

The convergence theorem for scheme (2.13) stated below is analogous to the corresponding proposition stated in [5] for a numerical scheme based on the Itô–Taylor expansion described in [17].

Let $M_2([0, T])$ be the class of stochastic functions $\xi(t, \omega) \overset{\text{def}}{=} \xi_t : [0, T] \times \Omega \longrightarrow \mathbb{R}^1$ that are measurable in all variables (t, ω). Let $\mathcal{F}_t$ be measurable at every $t \in [0, T]$ and satisfy the condition

$$\int\limits_0^T \mathbf{M}(\xi_t)^2 dt < \infty.$$

Define

$$\mathcal{A}_q = \left\{(k, j, l_1, \ldots, l_k) : k + j + \sum_{p=1}^{k} l_p = q;\ k, j, l_1, \ldots, l_k = 0, 1, \ldots\right\}. \tag{2.19}$$

Theorem 1. *Suppose that the following conditions are satisfied for certain* $\Delta > 0$, $r \in N$, *and* $\{\tau_j\}_{j=0}^{n_T+1}$ *and for every* $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$ *and* $\mathbf{y} \in X$ $(i_1, \ldots, i_k = 1, 2, \ldots, m)$:

(i) *for every* $(k, j, l_1, \ldots, l_k) \in \bigcup_{q=1}^{r} \mathfrak{D}_q$ *and* $t \in [0, T]$, *it holds that*

$$\left|G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{x} - G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{z}\right| \le K_1 |\mathbf{x} - \mathbf{z}|, \quad K_1 < \infty,$$

$$|\mathbf{c}(\mathbf{x}, t, \mathbf{y}) - \mathbf{c}(\mathbf{z}, t, \mathbf{y})| \le C_1(\mathbf{y})|\mathbf{x} - \mathbf{z}|, \quad |\mathbf{c}(\mathbf{x}, t, \mathbf{y})| \le C_2(\mathbf{y})(1 + |\mathbf{x}|);$$

(ii) *for every* $(k, j, l_1, \ldots, l_k) \in \bigcup_{q=1}^{r} \mathcal{A}_q$ *and* $t \in [0, T]$, *it holds that*

$$G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{x} \in \mathcal{L}, \quad G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{x}_{\tau_p} \in M_2([0, T]),$$

$$\left|G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{x}\right| \le K_2(1 + |\mathbf{x}|), \quad K_2 < \infty;$$

where $p = 0, 1, \ldots, n_T + 1$;

(iii) *for every* $(k, j, l_1, \ldots, l_k) \in \mathcal{A}_r$ *and* $t \in [0, T]$, *it holds that*

$$L G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{x}_t, \quad G_0^{(g)} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{x}_t \in M_2([0, T]),$$

$$\left|L G_1^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{x}\right| + \left|G_0^{(g)} G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} L^j \mathbf{x}\right| \le K_1(1 + |\mathbf{x}|),$$

where $g = 1, 2, \ldots, m$;

(iv) *for every* $s, t \in [0, T]$ *and* $i = 1, 2, \ldots, m$, *it holds that*

$$|\mathbf{a}(\mathbf{x}, s) - \mathbf{a}(\mathbf{x}, t)| \le K_3(1 + |\mathbf{x}|)|s - t|^{1/2}, \quad K_3 < \infty,$$

$$|B_i(\mathbf{x}, s) - B_i(\mathbf{x}, t)| \le K_4(1 + |\mathbf{x}|)|s - t|^{1/2}, \quad K_4 < \infty,$$

$$|\mathbf{c}(\mathbf{x}, s, \mathbf{y}) - \mathbf{c}(\mathbf{x}, t, \mathbf{y})| \le C_3(\mathbf{y})(1 + |\mathbf{x}|)|s - t|^{1/2},$$

where $(C_i(\mathbf{y}))^2$, $(i = 1, 2, 3)$ *are* $\Pi(d\mathbf{y})$-*integrable scalar functions.*

Then,

$$\mathbf{M}\left\{\sup_{0 \le t \le T} |\mathbf{x}_t - \bar{\mathbf{y}}_t|^2 \middle| \mathcal{F}_0\right\} \le K_5(1 + |\mathbf{x}_0|^2)\Delta^r + K_6|\mathbf{x}_0 - \bar{\mathbf{y}}_0|^2, \tag{2.20}$$

where the constants K_5 *and* K_6 *are independent of* Δ, *and*

$$\begin{aligned}\bar{\mathbf{y}}_t = \bar{\mathbf{y}}_0 + \sum_{q=1}^{r} \sum_{(k, j, l_1, \ldots, l_k) \in \mathfrak{D}_q} & \left[\sum_{p=0}^{n_t - 1} \frac{(\Delta_p)^j}{j!} {}^{(k)}G_{l_1} \ldots G_{l_k} L^j \{\bar{\mathbf{y}}_{\tau_p}\}^{(k)} I_{(l_1 \ldots l_k)\tau_{p+1}, \tau_p}\right. \\ & \left. + \frac{(t - \tau_{n_t})^j}{j!} {}^{(k)}G_{l_1} \ldots G_{l_k} L^j \{\bar{\mathbf{y}}_{\tau_{n_t}}\}^{(k)} I_{(l_1 \ldots l_k)t, \tau_{n_t}}\right] + \int_0^t \int_X \mathbf{c}(\bar{\mathbf{y}}_{\tau_{n_s}}, \tau_{n_s}, \mathbf{y}) \nu(ds, d\mathbf{y}),\end{aligned} \tag{2.21}$$

where $t \in [0, T]$ *and* $\bar{\mathbf{y}}_0$ *and* $\mathbf{x}_0$ *are* $\mathcal{F}_0$-*measurable random variables.*

Theorem 1 is the analogue of Theorem 3 in [5] adapted to scheme (2.13) under the assumption that $\{\tau_j\}_{j=0}^{n_T+1}$ is the discretization constructed by using the jump times of a Poisson process as described above.

Theorem 2. *Suppose that the following conditions are satisfied in addition to those formulated in Theorem* 1:

(i) $\mathbf{M}\{|\mathbf{x}_0|^2\} < \infty$, $\mathbf{M}\{|\mathbf{x}_0 - \bar{\mathbf{y}}_0|^2\} \le K_7\Delta^r$, *and* $\mathbf{M}\{|\mathbf{x}_0 - \mathbf{y}_0|^2\} \le K_8\Delta^r$, *where* K_7 *and* $K_8 < \infty$ *and* $\mathbf{y}_0$ *is an* $\mathcal{F}_0$-*measurable random variable;*

(ii) *for every* $p = 0, 1, \ldots, n_t - 1$, $i_1, \ldots, i_k = 1, 2, \ldots, m$; *and* $(k, j, l_1, \ldots, l_k) \in \bigcup_{q=1}^{r}\mathfrak{D}_q$, *it holds that*

$$\mathbf{M}\left\{(I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p} - \hat{I}^{(i_1\ldots i_k)}_{(l_1\ldots l_k)\tau_{p+1},\tau_p})^2\right\} \le C_1\Delta^{r+1},$$
$$\mathbf{M}\left\{(I^{(i_1\ldots i_k)}_{(l_1\ldots l_k)t,\tau_{n_t}} - \hat{I}^{(i_1\ldots i_k)}_{(l_1\ldots l_k)t,\tau_{n_t}})^2\right\} \le C_1\Delta^{r+1}, \tag{2.22}$$

where $C_1 < \infty$ *and* $t \in [0, T]$.

Then, $\mathbf{M}\{|\mathbf{x}_T - \mathbf{y}_T|^2\} \le C_2\Delta^r$, *where the constant* C_2 *is independent of* Δ *and the process* $\mathbf{y}_t$ $(t \in [0, T])$ *is obtained by replacing each MSI in the process* $\bar{\mathbf{y}}_t$ $(t \in [0, T])$ *defined by* (2.21) *by its approximation.*

Theorem 2 is analogous to the corollaries to Theorem 3 in [5].

Analogues of Theorems 1 and 2 can also be formulated for schemes (2.14), (2.17), and (2.18).

3. STRONG APPROXIMATION METHODS FOR STRATONOVICH AND ITÔ MSIS

To implement the numerical schemes constructed above, approximations of the following Stratonovich and Itô MSIs are required:

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}^{(i_1)}_{t_1}\ldots d\mathbf{w}^{(i_k)}_{t_k}, \tag{3.1}$$

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}^{(i_1)}_{t_1}\ldots d\mathbf{w}^{(i_k)}_{t_k}, \tag{3.2}$$

where every $\psi_l(\tau)$ $(l = 1, 2, \ldots, k)$ is a continuously differentiable function on $[t, T]$; $\mathbf{w}^{(i)}_t = \mathbf{f}^{(i)}_t$ for $i = 1, 2, \ldots, m$; $\mathbf{w}^{(0)}_t = t$, $i_1, \ldots, i_k = 0, 1, \ldots, m$; and $\int$ and $\int^*$ denote Itô and Stratonovich stochastic integrals, respectively. It is obvious that MSIs (3.1) and (3.2) are more general than those in (2.13), (2.14), (2.17), and (2.18).

Let us briefly review the existing strong approximation methods for integrals (3.1) and (3.2) that can be invoked to implement the strong numerical schemes (2.13), (2.14), (2.17), and (2.18).

In [9] (see also [10, 21]), Mil'shtein proposed to expand (3.1) in multiple series in terms of products of standardized Gaussian variables by representing the Wiener process as a trigonometric Fourier series [4]. To obtain the Mil'shtein expansion of (3.1), the Fourier components of the Wiener process $\mathbf{f}_t$ must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of (3.1) valid for an arbitrary multiplicity k. For this reason, only expansions of single, double, and triple integrals (3.1) were presented in [9, 10, 21].

An alternative strong approximation method was proposed for (3.1) in [12, 19, 20], where $J^*[\psi^{(k)}]_{T,t}$ was represented as a multiple stochastic integral of a certain discontinuous nonrandom function of k variables, and the function was then expressed as a multiple Fourier series in a complete system of continuously differentiable functions that are orthonormal in $L_2([t, T])$. As a result, a general multiple series expansion of (3.1) in terms of products of standardized Gaussian random variables was obtained in [12, 19, 20] for an arbitrary multiplicity k. Hereinafter, this method is referred to as the method of multiple Fourier series.

The method of multiple Fourier series is based on the theorem formulated below and proved in [12, 19, 20].

Define the following function on a hypercube $[t, T]^k$:

$$K(t_1, \ldots, t_k) = \begin{cases} \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}}, & k \geq 2, \\ \psi_1(t_1), & k = 1, \end{cases} \tag{3.3}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let $\{\varphi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of continuously differentiable functions in $L_2([t, T])$.

Define the class $H([t, T]) \subset L_2([t, T])$ of piecewise smooth on (t, T) and bounded on $[t, T]$ functions $f(x)$ for which the Fourier series $\sum_{j=0}^{\infty} C_j \varphi_j(x)$ with $C_j = \int_t^T f(x)\varphi_j(x)dx$ converges to $[f(x+0) + f(x-0)]/2$ at any interior point x in $[t, T]$, uniformly converges to $f(x)$ on any closed continuity interval of $f(x)$, and is convergent at $x = t$ and T (convergence of Fourier series is henceforth interpreted in the sense of the $L_2([t, T])$ norm).

Define

$$C_{j_{q-1}\ldots j_1}(t_q, \ldots, t_k) \stackrel{\text{def}}{=} \int\limits_{[t, T]^{q-1}} K(t_1, \ldots, t_k) \prod_{l=1}^{q-1} \varphi_{j_l}(t_l) dt_1 \ldots dt_{q-1}, \quad 2 \leq q \leq k,$$
$$C_{j_0\ldots j_1}(t_1, \ldots, t_k) \stackrel{\text{def}}{=} K(t_1, \ldots, t_k). \tag{3.4}$$

Theorem 3 (see [12, 19, 20]). *Suppose that the following conditions are satisfied for a certain* $k \geq 1$:

(i) *every* $\psi_i(\tau)$ $(i = 1, 2, \ldots, k)$ *is a continuously differentiable on* $[t, T]$ *function*;

(ii) $\{\varphi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of continuously differentiable functions in* $L_2([t, T])$;

(iii) $G_{j_{q-1}\ldots j_1}(t_q, \ldots, t_k) \in H([t, T])$, *as a function of* t_q $(q = 1, 2, \ldots, k)$.

Then, integral (3.1) *can be represented as the following multiple series that is convergent in the power n–mean norm* ($n \in N$):

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{(j_l)T,t}^{(i_l)}, \tag{3.5}$$

where $i_1, \ldots, i_k = 0, 1, \ldots, m$; *every*

$$\zeta_{(j_l)T,t}^{(i_l)} = \int_t^T \varphi_{j_l}(s) d\mathbf{w}_s^{(i_l)}$$

is a standardized Gaussian random variable with various i_l (*or* j_l *if* $i_l \neq 0$); *and*

$$C_{j_k\ldots j_1} = \int\limits_{[t, T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \varphi_{j_l}(t_l) dt_1 \ldots dt_k. \tag{3.6}$$

It was shown in [12] that the method of multiple Fourier series leads to the Mil'shtein expansion of (3.1) in the case of a trigonometric system of functions and to a substantially simpler expansion of (3.1) in the case of a polynomial system of functions.

In the available literature, strong approximations of (3.2) are constructed by rectangle or trapezoid methods (e.g., see [9]). These methods are characterized by several times lower mean-square convergence rates as compared to Mil'shtein's method [9] and the method of multiple Fourier series [12]. However, some well-known relations between integrals (3.1) and (3.2) (see [10, 12]) can be used to represent (3.2) as a finite sum of integrals of the form of (3.1) with multiplicities not greater than k, which can then be approximated by the method of multiple Fourier series.

Note that, if the integral $J^*[\psi^{(k)}]_{T,t}$ is approximated by

$$J^*[\psi^{(k)}]^q_{T,t} = \sum_{j_1, \ldots, j_k = 0}^{q} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta^{(i_l)}_{(j_l)T,t}, \tag{3.7}$$

where $q < \infty$ ($i_1, \ldots, i_k = 1, 2, \ldots, m$), and the indices are pairwise distinct, then (see [12, 20])

$$\mathbf{M}\{(J^*[\psi^{(k)}]^q_{T,t} - J^*[\psi^{(k)}]_{T,t})^2\} = \int\limits_{[t,T]^k} K^2(t_1, \ldots, t_k)dt_1 \ldots dt_k - \sum_{j_1, \ldots, j_k = 0}^{q} C^2_{j_k \ldots j_1}. \tag{3.8}$$

Approximations of (3.1) by multiple Fourier series in polynomial functions were presented in [12, 19, 20] for single and double integrals and in [22] for multiplicities running from 1 to 5. The mean-square errors were also calculated for the approximations of stochastic integrals considered in these publications.

As an application of Theorem 3, consider the case of a complete orthonormal polynomial system in $L_2([t, T])$:

$$\varphi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2}\right)\frac{2}{T-t}\right), \quad j = 0, 1, \ldots, \tag{3.9}$$

where $P_j(x)$ is a Legendre polynomial.

Applying Theorem 3 and using (3.9), one obtains

$$I^{*(i_1)}_{(0)T,t} = \sqrt{T-t}\,\zeta^{(i_1)}_0, \quad I^{*(i_1)}_{(1)T,t} = -\frac{(T-t)^{3/2}}{2}\left(\zeta^{(i_1)}_0 + \frac{1}{\sqrt{3}}\zeta^{(i_1)}_1\right) \text{ w. p. 1},$$

$$I^{*(i_1)}_{(2)T,t} = \frac{(T-t)^{5/2}}{3}\left(\zeta^{(i_1)}_0 + \frac{\sqrt{3}}{2}\zeta^{(i_1)}_1 + \frac{1}{2\sqrt{5}}\zeta^{(i_1)}_2\right) \text{ w. p. 1}, \tag{3.10}$$

$$I^{*(i_2 i_1)}_{(00)T,t} = \frac{T-t}{2}\left[\zeta^{(i_1)}_0\zeta^{(i_2)}_0 + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}(\zeta^{(i_2)}_{i-1}\zeta^{(i_1)}_i - \zeta^{(i_2)}_i\zeta^{(i_1)}_{i-1})\right],$$

$$I^{*(i_2 i_1)}_{(01)T,t} = -\frac{(T-t)^2}{4}\left\{\frac{4}{3}\zeta^{(i_2)}_0\zeta^{(i_1)}_0 + \frac{2}{\sqrt{3}}\zeta^{(i_2)}_0\zeta^{(i_1)}_1 + \frac{2}{3\sqrt{5}}\zeta^{(i_2)}_0\zeta^{(i_1)}_2\right.$$

$$-\frac{1}{3\sqrt{5}}\zeta^{(i_2)}_2\zeta^{(i_1)}_0 + \sum_{i=1}^{\infty}\left[\frac{1}{\sqrt{(2i+1)(2i+3)}}\zeta^{(i_2)}_i\zeta^{(i_1)}_{i+1} - \frac{1}{\sqrt{4i^2-1}}\zeta^{(i_2)}_i\zeta^{(i_1)}_{i-1}\right. \tag{3.11}$$

$$\left.\left.-\frac{1}{(2i-1)(2i+3)}\zeta^{(i_2)}_i\zeta^{(i_1)}_i + \frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}((i+2)\zeta^{(i_2)}_i\zeta^{(i_1)}_{i+2} - (i+1)\zeta^{(i_2)}_{i+2}\zeta^{(i_1)}_i)\right]\right\},$$

$$I^{*(i_2 i_1)}_{(10)T,t} = -\frac{(T-t)^2}{4}\left\{\frac{2}{3}\zeta^{(i_2)}_0\zeta^{(i_1)}_0 + \frac{1}{3\sqrt{5}}\zeta^{(i_1)}_2\zeta^{(i_2)}_0 - \frac{2}{3\sqrt{5}}\zeta^{(i_2)}_2\zeta^{(i_1)}_0 + \frac{1}{\sqrt{3}}\zeta^{(i_1)}_0\zeta^{(i_2)}_1 + \frac{1}{\sqrt{3}}\zeta^{(i_2)}_0\zeta^{(i_1)}_1\right.$$

$$+\sum_{i=1}^{\infty}\left[\frac{-1}{\sqrt{4i^2-1}}\zeta^{(i_2)}_i\zeta^{(i_1)}_{i-1} + \frac{1}{\sqrt{(2i+1)(2i+3)}}\zeta^{(i_2)}_i\zeta^{(i_1)}_{i+1}\right. \tag{3.12}$$

$$\left.\left.+\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}((i+1)\zeta^{(i_2)}_i\zeta^{(i_1)}_{i+2} - (i+2)\zeta^{(i_2)}_{i+2}\zeta^{(i_1)}_i) + \frac{1}{(2i-1)(2i+3)}\zeta^{(i_1)}_i\zeta^{(i_2)}_i\right]\right\},$$

where

$$\zeta_j^{(i)} = \int_t^T \varphi_j(s) d\mathbf{f}_s^{(i)},$$

$$
\begin{aligned}
I^{*(i_3 i_2 i_1)}_{(000)T,t} = & -\frac{1}{T-t}\left(I^{*(i_1)}_{(0)T,t} I^{*(i_2 i_3)}_{(10)T,t} + I^{*(i_3)}_{(0)T,t} I^{*(i_2 i_1)}_{(10)T,t}\right) + \frac{1}{2} I^{*(i_1)}_{(0)T,t}\left(I^{*(i_3 i_2)}_{(00)T,t} - I^{*(i_2 i_3)}_{(10)T,t}\right) \\
& + \frac{1}{6}(T-t)^{3/2}\zeta_0^{(i_1)}\left\{\zeta_0^{(i_3)}\left(\frac{1}{\sqrt{5}}\zeta_2^{(i_2)} - \sqrt{3}\zeta_1^{(i_2)} - \zeta_0^{(i_2)}\right) + \zeta_1^{(i_3)}\zeta_1^{(i_2)}\right\} \\
& - \frac{(T-t)^{3/2}}{4}\sum_{k=1}^{\infty}\Bigg[\sum_{\substack{j=0 \\ k+j-i = 2p \ge 0, p \in Z}}^{k+1} \sum_{i=0}^{\infty} L_{ijk} K_{j,k+1,(k+j-i)/2} \zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} \\
& + \sum_{\substack{j=k+2 \\ k+j-i = 2p \ge 0, p \in Z}}^{\infty} \sum_{i=0}^{\infty} L_{ijk} K_{k+1,j,(k+j-i)/2} \zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} \\
& - \Bigg(\sum_{\substack{j=0 \\ k+j-i = 2p \ge 2, p \in Z}}^{k-1} \sum_{i=0}^{\infty} L_{ijk} K_{j,k-1,(k+j-i)/2-1} \zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} \\
& + \sum_{\substack{j=k \\ k+j-i = 2p \ge 2, p \in Z}}^{\infty} \sum_{i=0}^{\infty} L_{ijk} K_{k-1,j,(k+j-i)/2-1} \zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} \\
& + \sum_{\substack{j=0 \\ k+j-i = 2p \ge -2, p \in Z}}^{k+1} \sum_{i=0}^{\infty} M_{ijk} K_{j,k+1,(k+j-i)/2+1} \zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} \\
& + \sum_{\substack{j=k+2 \\ k+j-i = 2p \ge -2, p \in Z}}^{\infty} \sum_{i=0}^{\infty} M_{ijk} K_{k+1,j,(k+j-i)/2+1} \zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}\Bigg) \\
& + \sum_{\substack{j=0 \\ k+j-i = 2p \ge 0, p \in Z}}^{k-1} \sum_{i=0}^{\infty} M_{ijk} K_{j,k-1,(k+j-i)/2} \zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} \\
& + \sum_{\substack{j=k \\ k+j-i = 2p \ge 0, p \in Z}}^{\infty} \sum_{i=0}^{\infty} M_{ijk} K_{k-1,j,(k+j-i)/2} \zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}\Bigg]
\end{aligned}
\tag{3.13}
$$

with

$$L_{ijk} = \sqrt{\frac{2j+1}{(2k+1)(2i+1)}}\frac{1}{2i+3}, \quad M_{ijk} = \sqrt{\frac{2j+1}{(2k+1)(2i+1)}}\frac{1}{2i-1},$$

$$K_{mnk} = \frac{a_{m-k}a_k a_{n-k}}{a_{m+n-k}}\frac{2n+2m-4k+1}{2n+2m-2k+1}, \quad a_k = \frac{(2k-1)!!}{k!}, \quad m \le n.$$

Expansions (3.10)–(3.13) converge to the respective MSIs in the mean–power n norm ($n \in N$).

According to (3.8), if $i_1 \neq i_2$, then

$$\mathbf{M}\left\{\left(I^{*(i_2 i_1)}_{(00)T,t} - I^{*(i_2 i_1)q}_{(00)T,t}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right),$$

$$\mathbf{M}\left\{\left(I^{*(i_2 i_1)}_{(10)T,t} - I^{*(i_2 i_1)q}_{(10)T,t}\right)^2\right\} = \mathbf{M}\left\{\left(I^{*(i_2 i_1)}_{(01)T,t} - I^{*(i_2 i_1)q}_{(01)T,t}\right)^2\right\}$$

$$= \frac{(T-t)^4}{16}\left\{\frac{7}{9} - \sum_{i=1}^{q}\left(\frac{1}{4i^2-1} + \frac{1}{(2i+1)(2i+3)} + \frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2} + \frac{1}{(2i-1)^2(2i+3)^2}\right)\right\},$$

where $I^{*(i_2 i_1)q}_{(00)T,t}$, $I^{*(i_2 i_1)q}_{(10)T,t}$, and $I^{*(i_2 i_1)q}_{(01)T,t}$ are obtained by replacing ∞ with q on the right-hand sides of (3.10)–(3.13).

4. WEAK NUMERICAL METHODS FOR JUMP–DIFFUSION SDES

Weak numerical methods for (2.1) are constructed here in the autonomous case, i.e., when $\mathbf{a}(\mathbf{x}, t) \equiv \mathbf{a}(\mathbf{x})$, $\mathbf{B}(\mathbf{x}, t) \equiv \mathbf{B}(\mathbf{x})$, and $\mathbf{c}(\mathbf{x}, t, \mathbf{y}) \equiv \mathbf{c}(\mathbf{x}, \mathbf{y})$. The weak numerical methods are based on (2.6) and the unified Itô–Taylor expansions described in [12, 15]:

$$\begin{cases} \mathbf{y}_{\tau_{p+1}-} = \mathbf{y}_{\tau_p} + \sum\limits_{q=1}^{r} \sum\limits_{(k, j, l_1, \dots, l_k) \in \mathcal{A}_q} \dfrac{(\Delta_p)^j}{j!} {}^{(k)}\tilde{G}_{l_1}\dots\tilde{G}_{l_k}\tilde{L}^j\{\mathbf{y}_{\tau_p}\} \,\dot{}\, {}^{k\,(k)}\hat{I}_{(l_1\dots l_k)\tau_{p+1},\tau_p}, \\ \mathbf{y}_{\tau_{p+1}} = \mathbf{y}_{\tau_{p+1}-} + \int\limits_X \mathbf{c}(\mathbf{y}_{\tau_{p+1}-}, \tau_{p+1}, \mathbf{y})\nu(\{\tau_{p+1}\}, d\mathbf{y}), \end{cases} \tag{4.1}$$

$$\begin{cases} \mathbf{y}_{\tau_{p+1}-} = \mathbf{y}_{\tau_p} + \sum\limits_{q=1}^{r} \sum\limits_{(k, j, l_1, \dots, l_k) \in \mathcal{A}_q} \dfrac{(\Delta_p)^j}{j!} {}^{(k)}\tilde{L}^j\tilde{G}_{l_1}\dots\tilde{G}_{l_k}\{\mathbf{y}_{\tau_p}\} \,\dot{}\, {}^{k\,(k)}\hat{J}_{(l_1\dots l_k)\tau_{p+1},\tau_p}, \\ \mathbf{y}_{\tau_{p+1}} = \mathbf{y}_{\tau_{p+1}-} + \int\limits_X \mathbf{c}(\mathbf{y}_{\tau_{p+1}-}, \tau_{p+1}, \mathbf{y})\nu(\{\tau_{p+1}\}, d\mathbf{y}), \end{cases} \tag{4.2}$$

where $p = 0, 1, \dots, n_T$, $\tau_{p+1} - \tau_p \overset{\text{def}}{=} \Delta_p$, $\mathcal{A}_q$ is defined by (2.19); $\tilde{G}^{(i)}_q = (\tilde{G}^{(i)}_{q-1}\tilde{L} - \tilde{L}\tilde{G}^{(i)}_{q-1})/q$, where $q = 1, 2, \dots$, and $i = 1, 2, \dots, m$; the operators $\tilde{L}$ and $\tilde{G}^{(i)}_0$ $(i = 1, 2, \dots, m)$ are obtained by replacing $\mathbf{a}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$, and $\mathbf{c}(\mathbf{x}, t, \mathbf{y})$ with $\mathbf{a}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, and $\mathbf{c}(\mathbf{x}, \mathbf{y})$ in L and $G^{(i)}_0$ $(i = 1, 2, \dots, m)$, respectively; and the remaining notation is similar to that used in (2.13) and (2.14). The partial derivatives and MSIs on the right-hand sides of (4.1) and (4.2) are supposed to exist.

Numerical schemes (4.1) and (4.2) substantially differ from (2.13) and (2.14) in two respects. First, the right-hand sides in (4.1) and (4.2) contain all Itô MSIs involved in the unified Itô–Taylor expansions with multiplicities running from 1 to r [12], whereas (2.13) and (2.14) contain only the Itô MSIs that are $O((\tau_{p+1} - \tau_p)^\gamma)$ in the mean-square sense as $\tau_{p+1} \downarrow \tau_p$ $(\gamma = 0.5, 1.0, 1.5, \dots, 0.5r)$. Second, the Itô MSIs in (2.13) and (2.14) are approximated in the strong sense defined by (2.22), whereas the Itô MSIs in (4.1) and (4.2) are approximated in the weak sense defined by condition (4.3) below.

Denote by $\tilde{\mathbf{y}}_t$ $(t \in [0, T])$ the random process obtained by replacing the operators $G_{l_1}, \dots, G_{l_k}$, and L in (2.21) with $\tilde{G}_{l_1}, \dots, \tilde{G}_{l_k}$, and $\tilde{L}$ and the set $\mathfrak{D}_q$ with $\mathcal{A}_q$, respectively.

Let $C^l(\mathbb{R}^n, \mathbb{R}^1)$ be the space of l times continuously differentiable functions $g : \mathbb{R}^n \longrightarrow \mathbb{R}^1$ that have polynomial growth, including their partial derivatives to order l. (A function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^1$ has polynomial growth if there exist constants $C_F > 0$ and $\gamma_F \in N$ that depend on F, and $|F(\mathbf{x})| \le C_F(1 + |\mathbf{x}|^{2\gamma_F})$ for every $x \in \mathbb{R}^n$.)

The weak convergence theorem formulated here for a discrete approximation $\tilde{\mathbf{y}}_{\tau_p}$ $(p = 0, 1, \dots, n_T + 1)$ is an analogue of the corresponding proposition about the weak convergence of a numerical scheme similar to (4.1), but based on the Itô–Taylor expansion [17], which was stated in [7].

Theorem 4. *Suppose that the following conditions are satisfied for a certain* $r \in N$ *and a function* $g(\mathbf{x})$: $\mathbb{R}^n \longrightarrow \mathbb{R}^1$:

(i) $g(\mathbf{x}) \in C^{2(r+1)}(\mathbb{R}^n, \mathbb{R}^1)$;

(ii) $\mathbf{a}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, *and* $\mathbf{c}(\mathbf{x}, \mathbf{y})$ *are* $2(r + 1)$ *times continuously differentiable functions with bounded derivatives*;

(iii) *for every* $(k, j, l_1, \dots, l_k) \in \bigcup_{q=1}^{r} \mathcal{A}_q$, *and* $\mathbf{x} \in \mathbb{R}^n$, *it holds that*

$$\left|\tilde{G}_{l_1}^{(i_1)} \dots \tilde{G}_{l_k}^{(i_k)} \tilde{L}^j \mathbf{x}\right| \le K(1 + |\mathbf{x}|), \quad K < \infty,$$

where $i_1, \dots, i_k = 1, 2, \dots, m$.

Then, $|\mathbf{M}\{g(\mathbf{x}_T)\} - \mathbf{M}\{g(\tilde{\mathbf{y}}_T)\}| \le C\Delta^r$, *where the constant* $C < \infty$ *is independent of* Δ.

According to [7, 10], if the assumptions of Theorem 4 are supplemented by the condition $(k_g, j, l_1^{(g)}, \dots, l_{k_g}^{(g)}) \in \mathcal{A}_{k_g}$, $i_1^{(g)}, \dots, i_{k_g}^{(g)} = 1, 2, \dots, m$, $k_g \le r$, $g = 1, 2, \dots, l$, $l = 1, 2, \dots, 2r + 1$, $p = 0, 1, \dots, n_T$,

$$\left|\mathbf{M}\left\{\prod_{g=1}^{l} I_{(l_1^{(g)} \dots l_{k_g}^{(g)})\tau_{p+1}, \tau_p}^{(i_1^{(g)} \dots i_{k_g}^{(g)})} - \prod_{g=1}^{l} \hat{I}_{(l_1^{(g)} \dots l_{k_g}^{(g)})\tau_{p+1}, \tau_p}^{(i_1^{(g)} \dots i_{k_g}^{(g)})} \middle| \mathcal{F}_{\tau_p}\right\}\right| \le K_1 \Delta^{r+1-j}, \tag{4.3}$$

where $K_1 < \infty$ is constant, then

$$|\mathbf{M}\{g(\mathbf{x}_T)\} - \mathbf{M}\{g(\mathbf{y}_T)\}| \le C_1 \Delta^r, \tag{4.4}$$

where the constant $C_1 < \infty$ is independent of Δ and $\mathbf{y}_T$ is determined by numerical scheme (4.1) with $p = n_T$.

Condition (4.3) is analogous to the corresponding condition in [10] for MSIs in the Itô–Taylor expansion [17].

5. WEAK APPROXIMATION OF ITÔ MSIS

Note the following characteristics of the weak approximation of MSIs. First, the weak approximations of MSIs are substantially simpler [9, 10] as compared to their strong approximations and typically involve one, two, or three random variables with well-defined properties (cf. Theorem 3). This is explained by the fact that condition (4.3) leaves much more freedom in constructing approximations, as compared to (2.22). Second, in contrast to the strong approximations of MSIs (recall Theorem 3), their weak approximations cannot be constructed by applying any general "prescription." This means that the weak approximation of an Itô MSI should be constructed through a skilful choice so that condition (4.3) is satisfied for a given r. Generally, most of the weak approximations corresponding to a certain r cannot be used as weak approximations for $r_1 > r$. Note that strong approximations of MSIs have the converse property, because the variation of r affects only the number q in (3.7), whereas the general structure of the expression that approximates an MSI remains the same.

In [9, 10], weak approximations of the Itô MSIs involved in the Itô–Taylor expansion [17] were obtained for $r = 1$ and 2 in the case of a multidimensional Wiener process $\mathbf{f}_s$ and for $r = 3$ in the case of a scalar Wiener process.

The construction of weak approximations of Itô MSIs for $r > 3$ presents certain difficulties, because condition (4.3) for various $k, j, l_1, \dots, l_k, l, i_1, \dots, i_k$ involves tens of conditions for moments that must be satisfied. The number of distinct Itô MSIs that require weak approximation and the corresponding number of moment conditions can be drastically reduced by using the unified Itô–Taylor expansion [12] instead of the Itô–Taylor expansion [17]. In particular, when $r = 4$ and f_s is a scalar Wiener process, the right-hand side of (4.1) contains 15 distinct approximations of Itô MSIs, whereas the analogue of this numerical scheme based on the Itô–Taylor expansion [17] contains 26 distinct approximations of Itô MSIs. For this reason, no weak approximations of the MSIs were obtained for $r = 4$ in [9, 10].

According to (4.3), to construct weak approximations of the Itô MSIs for $r = 4$, the following moment relations must be taken onto account:

$$\mathbf{M}\{I_3\} = \mathbf{M}\{I_3(I_0)^2\} = \mathbf{M}\{I_3I_{00}\} = 0, \quad \mathbf{M}\{I_3I_0\} = -\Delta^4/4,$$

$$\mathbf{M}\{I_2(I_0)^2\} = \mathbf{M}\{I_2I_{00}\} = \mathbf{M}\{I_2I_{000}\} = \mathbf{M}\{I_2I_{0000}\} = 0,$$

$$\mathbf{M}\{I_2(I_{00})^2\} = \mathbf{M}\{I_2(I_0)^4\} = \mathbf{M}\{I_2I_{000}I_0\} = 0,$$

$$\mathbf{M}\{I_2I_{00}(I_0)^2\} = \mathbf{M}\{I_2I_{10}\} = \mathbf{M}\{I_2I_{01}\} = \mathbf{M}\{I_2I_1I_0\} = \mathbf{M}\{I_2\} = 0,$$

$$\mathbf{M}\{I_2I_0\} = \Delta^3/3, \quad \mathbf{M}\{I_2(I_0)^3\} = \Delta^4, \quad \mathbf{M}\{I_2I_{00}I_0\} = \Delta^4/3, \quad \mathbf{M}\{I_2I_1\} = -\Delta^4/4,$$

$$\mathbf{M}\{I_\mu\} = \mathbf{M}\{I_\mu I_0\} = \mathbf{M}\{I_\mu I_{000}\} = \mathbf{M}\{I_\mu(I_0)^3\} = 0,$$

$$\mathbf{M}\{I_\mu I_{00}I_0\} = \mathbf{M}\{I_\mu I_1\} = 0, \quad \mathbf{M}\{I_{20}(I_0)^2\} = \Delta^4/6, \quad \mathbf{M}\{I_{20}I_{00}\} = \Delta^4/12,$$

$$\mathbf{M}\{I_{11}(I_0)^2\} = \Delta^4/4, \quad \mathbf{M}\{I_{11}I_{00}\} = \Delta^4/8, \quad \mathbf{M}\{I_{02}(I_0)^2\} = \Delta^4/2,$$

$$\mathbf{M}\{I_{02}I_{00}\} = \Delta^4/4, \quad \mathbf{M}\{I_\lambda I_{01}\} = \mathbf{M}\{I_\lambda I_1I_0\} = 0,$$

$$\mathbf{M}\{I_\lambda\} = \mathbf{M}\{I_\lambda I_0\} = \mathbf{M}\{I_\lambda(I_0)^2\} = \mathbf{M}\{I_\lambda I_{00}\} = 0,$$

$$\mathbf{M}\{I_\lambda I_1\} = \mathbf{M}\{I_\lambda I_{0000}\} = \mathbf{M}\{I_\lambda(I_{00})^2\} = \mathbf{M}\{I_\lambda(I_0)^4\} = 0,$$

$$\mathbf{M}\{I_\lambda I_{000}I_0\} = \mathbf{M}\{I_\lambda I_{00}(I_0)^2\} = \mathbf{M}\{I_\lambda I_{10}\} = 0,$$

$$\mathbf{M}\{I_{100}I_{000}\} = -\Delta^4/24, \quad \mathbf{M}\{I_{100}(I_0)^3\} = -\Delta^4/4, \quad \mathbf{M}\{I_{100}I_{00}I_0\} = -\Delta^4/8,$$

$$\mathbf{M}\{I_{010}I_{000}\} = -\Delta^4/12, \quad \mathbf{M}\{I_{010}(I_0)^3\} = -\Delta^4/2, \quad \mathbf{M}\{I_{010}I_{00}I_0\} = -\Delta^4/4,$$

$$\mathbf{M}\{I_{001}I_{000}\} = -\Delta^4/8, \quad \mathbf{M}\{I_{001}(I_0)^3\} = -3\Delta^4/4, \quad \mathbf{M}\{I_{001}I_{00}I_0\} = -3\Delta^4/8,$$

$$\mathbf{M}\{I_\rho I_0\} = \mathbf{M}\{I_\rho I_{000}\} = \mathbf{M}\{I_\rho(I_0)^3\} = \mathbf{M}\{I_\rho I_{00}I_0\} = 0,$$

$$\mathbf{M}\{I_\rho I_1\} = \mathbf{M}\{I_\rho I_{0000}\} = \mathbf{M}\{I_\rho(I_0)^5\} = \mathbf{M}\{I_\rho(I_{00})^2I_0\} = 0,$$

$$\mathbf{M}\{I_\rho I_{00}(I_0)^3\} = \mathbf{M}\{I_\rho I_{000}(I_0)^2\} = \mathbf{M}\{I_\rho I_{0000}I_0\} = 0,$$

$$\mathbf{M}\{I_\rho I_{000}I_{00}\} = \mathbf{M}\{I_\rho I_{100}\} = \mathbf{M}\{I_\rho I_{010}\} = 0,$$

$$\mathbf{M}\{I_\rho I_{001}\} = \mathbf{M}\{I_\rho I_2\} = \mathbf{M}\{(I_\rho)^2I_0\} = \mathbf{M}\{I_\rho I_{00}I_1\} = 0,$$

$$\mathbf{M}\{I_{10}I_{01}I_0\} = \mathbf{M}\{I_\rho\} = \mathbf{M}\{I_\rho I_1(I_0)^2\} = 0,$$

$$\mathbf{M}\{I_{10}(I_0)^2\} = -\Delta^3/3, \quad \mathbf{M}\{I_{10}I_{00}\} = -\Delta^3/6, \quad \mathbf{M}\{I_{10}(I_{00})^2\} = -\Delta^4/3,$$

$$\mathbf{M}\{I_{10}(I_0)^4\} = -2\Delta^4, \quad \mathbf{M}\{I_{10}I_{000}I_0\} = -\Delta^4/6, \quad \mathbf{M}\{I_{10}I_{00}(I_0)^2\} = -5\Delta^4/6,$$

$$\mathbf{M}\{(I_{10})^2\} = \Delta^4/12, \quad \mathbf{M}\{I_{10}I_{01}\} = \Delta^4/8, \quad \mathbf{M}\{I_{10}I_1I_0\} = -5\Delta^4/24,$$

$$\mathbf{M}\{I_{01}(I_0)^2\} = -2\Delta^3/3, \quad \mathbf{M}\{I_{01}I_{00}\} = -\Delta^3/3, \quad \mathbf{M}\{I_{01}(I_{00})^2\} = -2\Delta^4/3,$$

$$\mathbf{M}\{I_{01}(I_0)^4\} = -4\Delta^4, \quad \mathbf{M}\{I_{01}I_{000}I_0\} = -\Delta^4/3, \quad \mathbf{M}\{I_{01}I_{00}(I_0)^2\} = -5\Delta^4/3,$$

$$\mathbf{M}\{(I_{01})^2\} = \Delta^4/4, \quad \mathbf{M}\{I_{01}I_1I_0\} = 3\Delta^4/8,$$

where $\mu \overset{\text{def}}{=} 02$, 11, or 20; $\lambda \overset{\text{def}}{=} 100$, 010, or 001; $\rho \overset{\text{def}}{=} 10$ or 01; $\mathbf{M}\{\cdot|\mathcal{F}_{\tau_p}\} \overset{\text{def}}{=} \mathbf{M}\{\cdot\}$; and $\hat{I}_{(l_1 \dots l_k)\tau_{p+1}, \tau_p} \overset{\text{def}}{=} \hat{I}_{l_1 \dots l_k}$ is the approximation of an Itô integral defined as

$$I_{(l_1 \dots l_k)\tau_{p+1}, \tau_p} = \int_{\tau_p}^{\tau_{p+1}} (\tau_p - t_1)^{l_k} \dots \int_{\tau_p}^{t_{k-1}} (\tau_p - t_k)^{l_1} df_{t_k} \dots df_{t_1},$$

where f_s is the standardized scalar Wiener process and $\tau_{p+1} - \tau_p \overset{\text{def}}{=} \Delta$.

These moment relations were obtained by invoking standard properties of Itô integrals [4] and the following relations implied by the Itô formula, which are valid with probability 1:

$$(I_0)^4 = 24I_{0000} + 12\Delta I_{00} + 3\Delta^2, \quad (I_{00})^2 = 6I_{0000} + 2\Delta I_{00} + \Delta^2/2,$$

$$I_{00}(I_0)^2 = 12I_{0000} + 5\Delta I_{00} + \Delta^2, \quad (I_0)^5 = 120I_{00000} + 60\Delta I_{000} + 15\Delta^2 I_0,$$

$$(I_{00})^2 I_0 = 30I_{00000} + 12\Delta I_{000} + 10\Delta^2 I_0/4,$$

$$I_{00}(I_0)^3 = 60I_{00000} + 27\Delta I_{000} + 6\Delta^2 I_0, \quad I_{00}I_1 = I_{001} + I_{010} + I_{100} - \Delta^2 I_0/2,$$

$$I_{000}(I_0)^2 = 20I_{00000} + 7\Delta I_{000} + \Delta^2 I_0, \quad I_{0000}I_0 = 5I_{00000} + \Delta I_{000},$$

$$I_{000}I_{00} = 10I_{00000} + 3\Delta I_{000} + \Delta^2 I_0/2,$$

$$I_{01}I_0 = 2I_{001} + I_{010} - (I_2 - \Delta^2 I_0)/2,$$

$$I_{10}I_0 = I_{010} + I_{100} + \Delta I_1 + I_2, \quad I_1 I_0 = I_{10} + I_{01} - \Delta^2/2,$$

$$I_{000}I_0 = 4I_{0000} + \Delta I_{00}, \quad (I_0)^2 = 2I_{00} + \Delta,$$

$$(I_0)^3 = 6I_{000} + 3\Delta I_0, \quad I_{00}I_0 = 3I_{000} + \Delta I_0,$$

where $I_{(l_1 \dots l_k)\tau_{p+1}, \tau_p} \overset{\text{def}}{=} I_{l_1 \dots l_k}$.

It is clear that condition (4.3) is satisfied for $r = 4$ if

$$\hat{I}_0 = \sqrt{\Delta}\zeta_0, \quad \hat{I}_{00} = \frac{1}{2}\Delta[(\zeta_0)^2 - 1], \quad \hat{I}_1 = -\frac{\Delta^{3/2}}{2}\left(\zeta_0 + \frac{1}{\sqrt{3}}\zeta_1\right),$$

$$\hat{I}_{000} = \Delta^{3/2}/6[(\zeta_0)^3 - 3\zeta_0], \quad \hat{I}_{0000} = \frac{\Delta^2}{24}[(\zeta_0)^4 + 6(\zeta_0)^2 + 3],$$

$$\hat{I}_{100} = -\frac{\Delta^{5/2}}{24}[(\zeta_0)^3 - 3\zeta_0], \quad \hat{I}_{010} = -\frac{\Delta^{5/2}}{12}[(\zeta_0)^3 - 3\zeta_0],$$

$$\hat{I}_{001} = -\frac{\Delta^{5/2}}{8}[(\zeta_0)^3 - 3\zeta_0], \quad \hat{I}_{11} = \frac{\Delta^3}{8}[(\zeta_0)^2 - 1],$$

$$\hat{I}_{20} = \frac{\Delta^3}{12}[(\zeta_0)^2 - 1], \quad \hat{I}_{02} = \frac{\Delta^3}{4}[(\zeta_0)^2 - 1],$$

$$\hat{I}_3 = -\frac{\Delta^{7/2}}{4}\zeta_0, \quad \hat{I}_2 = \frac{\Delta^{5/2}}{3}\left(\zeta_0 + \frac{\sqrt{3}}{2}\zeta_1\right),$$

$$\hat{I}_{10} = \Delta^2\left\{-\frac{1}{6}[(\zeta_0)^2 - 1] - \frac{1}{4\sqrt{3}}\zeta_0\zeta_1 \pm \frac{1}{12\sqrt{2}}[(\zeta_1)^2 - 1]\right\},$$

$$\hat{I}_{01} = \Delta^2\left\{-\frac{1}{3}[(\zeta_0)^2-1]-\frac{1}{4\sqrt{3}}\zeta_0\zeta_1 \mp \frac{1}{12\sqrt{2}}[(\zeta_1)^2-1]\right\}$$

with

$$\zeta_0 \stackrel{\text{def}}{=} \frac{1}{\sqrt{\Delta}}\int_{\tau_p}^{\tau_{p+1}} df_s, \quad \zeta_1 \stackrel{\text{def}}{=} \frac{2\sqrt{3}}{\Delta^{3/2}}\int_{\tau_p}^{\tau_{p+1}}\left(s-\tau_p-\frac{\Delta}{2}\right)df_s.$$

Here, ζ_0 and ζ_1 are independent standardized Gaussian random variables, and $\tau_{p+1}-\tau_p \stackrel{\text{def}}{=} \Delta$.

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REFERENCES

1. Shiryaev, A.N., *Osnovy stokhasticheskoi finansovoi matematiki* (Fundamentals of Stochastic Financial Mathematics), 2 vols., Moscow: Fazis, 1998.
2. Wright, D.J., Digital Simulation of Poisson Stochastic Differential Equations, *Int. J. Systems Sci.*, 1980, vol. 11, no. 6, pp. 781–785.
3. Dzagnidze, Z.A. and Chitashvili, R.Ya., Priblizhennoe integrirovanie stokhasticheskikh differentsial'nykh uravnenii, *Tr. Tbilisskogo Gos. Univ., Inst. Prikl. Mat.*, 1975, vol. IV, pp. 267–279.
4. Gikhman, I.I. and Skorokhod, A.V., *Teoriya sluchainykh protsessov* (Theory of Stochastic Processes), Moscow: Nauka, 1975, vol. 3.
5. Platen, E., An Approximation Method for a Class of Itô Processes with Jump Component, *Lietuvos Mat. Rink.*, 1982, vol. 22, pp. 124–136.
6. Maghsoodi, Y. and Harris, C.J., In-Probability Approximation and Simulation of Nonlinear Jump-Diffusion SDEs, *IMA J. Math. Control Inform.*, 1987, no. 4, pp. 65–92.
7. Mikulevicius, R. and Platen, E., Time Discrete Taylor Approximations for Itô Processes with Jump Component, *Math. Nachr.*, 1988, vol. 138, pp. 93–104.
8. Maghsoodi, Y., Mean-Square Efficient Numerical Solution of Jump-Diffusion SDEs, *Preprint of Univ. Southampton, UK*, 1994, no. OR72.
9. Mil'shtein, G.N., *Chislennoe integrirovanie stokhasticheskikh differentsial'nykh uravnenii* (Numerical Integration of Stochastic Differential Equations), Sverdlovsk: Ural. Univ., 1988.
10. Kloeden, P.E. and Platen, E., *Numerical Solution of Stochastic Differential Equations*, Berlin: Springer, 1992.
11. Kloeden, P.E., Platen, E., and Schurz, H., *Numerical Solution of SDEs through Computer Experiments*, Berlin: Springer, 1994.
12. Kuznetsov, D.F., *Chislennoe modelirovanie stokhasticheskikh differentsial'nykh uravnenii i stokhasticheskikh integralov* (Numerical Simulation of Stochastic Differential Equations and Stochastic Integrals), St. Petersburg: Nauka, 1999.
13. Platen, E., An Introduction to Numerical Methods for Stochastic Differential Equations, *Acta Numer.*, 1999, no. 8, pp. 195–244.
14. Nikitin, N.N. and Razevig, V.D., Digital Simulation Methods for Stochastic Differential Equations and Estimation of Their Errors, *Zh. Vychisl. Mat. Mat. Fiz.*, 1978, vol. 18, no. 1, pp. 106–117.
15. Kul'chitskii, O.Yu. and Kuznetsov, D.F., Unified Itô–Taylor Expansion, *Zap. Nauchn. Semin. POMI, Ross. Akad. Nauk, Veroyatn. Stat. 2*, 1997, vol. 244, pp. 186–204.
16. Kuznetsov, D.F., *Dva novykh predstavleniya razlozheniya Teilora-Stratonovicha* (Two New Representations of the Stratonovich–Taylor Expansion), St. Petersburg: SPb. Gos. Tekh. Univ., 1999.
17. Platen, E. and Wagner, W., On a Taylor Formula for a Class of Itô Processes, *Probl. Math. Stat.*, 1982, vol. 3, pp. 37–51.
18. Kloeden, P.E. and Platen, E., The Stratonovich- and Itô-Taylor Expansions, *Math. Nachr.*, 1991, vol. 151, pp. 33–50.
19. Kuznetsov, D.F., Applications of Approximation Methods for Stratonovich and Itô Multiple Stochastic Integrals in Numerical Simulation of Controlled Stochastic Systems, *Probl. Upravl. Informatiki*, 1999, no. 4, pp. 91–108.
20. Kuznetsov, D.F., Expansion of Stratonovich Multiple Stochastic Integrals Based on Multiple Fourier Series, *Zap. Nauchn. Semin. POMI, Ross. Akad. Nauk, Veroyatn. Stat. 3*, 1999, vol. 260, pp. 164–185.
21. Kloeden, P.E., Platen, E., and Wright, I.W., The Approximation of Multiple Stochastic Integrals, *Stoch. Analys. Appl.*, 1992, vol. 10, no. 4, pp. 431–441.
22. Kuznetsov, D.F., *Primenenie polinomov Lezhandra k sil'noi approksimatsii reshenii stokhasticheskikh differentsial'nykh uravnenii* (Applications of Legendre Polynomials in Strong Approximations of Solutions to Stochastic Differential Equations), St. Petersburg: SPb. Gos. Tekh. Univ., 1999.

ПОПРАВКА

В статье Д.Ф. Кузнецова "Новые представления явных шаговых методов для стохастических дифференциальных уравнений со скачкообразной компонентой", опубликованной в № 6 нашего журнала с. г., допущены следующие опечатки:

стр. 926, формулы (2.17), (2.18): везде вместо Δp должно быть Δ_p;

стр. 930–931: номер формулы (3.10) относится к первой строке на стр. 931;

стр. 931, формула (3.13): вместо $I^{*(i_2 i_3)}_{(10)T,t}$ должно быть $I^{*(i_2 i_3)}_{(00)T,t}$;

стр. 935, 11-я строка сверху: величина $5\Delta^4/24$ положительная;

3-я строка снизу: вместо $(I_2 - \Delta^4 I_0)/2$ должно быть $(I_2 + \Delta^2 I_0)/2$;

стр. 936, 5-я строка сверху: должно быть

$$\hat{I}_{000} = \frac{\Delta^{3/2}}{6}[(\zeta_0)^3 - 3\zeta_0], \quad \hat{I}_{0000} = \frac{\Delta^2}{24}[(\zeta_0)^4 - 6(\zeta_0)^2 + 3].$$

Редакция