The Three-Step Strong Numerical Methods of the Orders of Accuracy 1.0 and 1.5 for Ito Stochastic Differential Equations

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This work deals with the three-step strong numerical methods of the orders of accuracy 1.0 and 1.5 for Ito stochastic differential equations. We propose the explicit and implicit three-step numerical schemes, as well as the three-step numerical schemes of a Runge–Kutta type. The convergence of the considered numerical methods is proved theoretically and illustrated by numerical experiments.

Key words: strong numerical method, multi-step method, the order of accuracy, Ito stochastic differential equation, convergence.

Assume that $(\Omega, \mathcal{F}, \mathrm{P})$ is a complete probabilistic space, $\{\mathcal{F}_t,\ t \in [0, T]\}$ is a non-decreasing right continuous family of σ-subalgebras $\mathcal{F}$. Let us consider the Ito stochastic differential equation in an integral notation

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{a}(\mathbf{x}_s, s)\, ds + \int_0^t B(\mathbf{x}_s, s)\, d\mathbf{w}_s, \tag{1}$$

here $\mathbf{x}_s \in \mathcal{R}^n$ is a strong solution of equation (1); $\mathbf{w}_s \in \mathcal{R}^m$ is a $\mathcal{F}_s$-measurable for every $s \in [0, T]$ standard Wiener process with the independent components $\mathbf{w}_s^{(i)}$ $(i = 1,\ldots, m)$; $\mathbf{x}_0 \in \mathcal{R}^n$ is an initial condition which does not depend stochastically upon the increments $\mathbf{w}_s - \mathbf{w}_0$ when $s > 0$, at that $\mathbf{M}\{|\ \mathbf{x}_0|^2\} < \infty$; the non random functions $\mathbf{a}(\mathbf{x}, s) : \mathcal{R}^n \times [0, T] \to \mathcal{R}^n$, $B(\mathbf{x}, s) : \mathcal{R}^n \times [0, T] \to \mathcal{R}^{n\times m}$ satisfy the conditions of existence and uniqueness, in the sense of stochastic equivalence, of a strong solution of equation (1) [1].

In recent years considerable growth of a number of works which deal with numerical integration of stochastic differential equations has been registered. It is related with the fact that these equations are adequate mathematical models of a number of physical and engineering systems [2], just as they are applied while solving such mathematical problems as the filtering problem, parameterization procedures testing problem, Cauchy problem for partial differential equations of the parabolic type and others [2].

In this work we consider the five- and eight-parameter families of three-step strong numerical methods of the orders of accuracy 1.0 and 1.5 correspondingly for Ito stochastic differential equations.

We shall remind a definition of the strong numerical method [2].

Let us consider the partitioning $\{\tau_p\}_{p=0}^{N}$ of the interval $[0, T]$ such that

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T,\ \Delta_N = \max_{0\le p\le N-1} |\tau_{p+1} - \tau_p|.$$

We shall say that the discrete approximation $\mathbf{y}_{\tau_p}$, $p = 0, 1,\ldots, N$, of the process $\mathbf{x}_t$, $t \in [0, T]$, which corresponds to the maximum step of digitization Δ_N, converges strongly with the order $\gamma > 0$ at the instant of time T to the process $\mathbf{x}_t$, $t \in [0, T]$, when $\Delta_N \downarrow 0$, if there exist the constant $C > 0$, which does not depend upon Δ_N, and number $\delta > 0$ such that $\mathbf{M}\{|\mathbf{x}_T - \mathbf{y}_T)|\} \le C\,(\Delta_N)^{\gamma}\ \forall\ \Delta_N \in (0, \delta)$.

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Advisability of construction of three-step strong numerical methods for Ito stochastic differential equations is caused by two reasons. First, it is well known [2, 3], that in a number of cases the use of multi-step numerical methods both for ordinary and stochastic differential equations enables, on every step of integration, to compute or approximate the less number of partial derivatives from the right-hand part of the considered equation as against one-step numerical methods of the same orders of accuracy being applied. Secondly, only a few works deal with multi-step numerical methods for Ito stochastic differential equations, that is why the given numerical methods are insufficiently studied. At that in the literature among multi-step methods only the two-step numerical ones for Ito stochastic differential equations are presented.

So, in [4] the explicit two-step strong numerical method of the order of accuracy 1.0 for the system of two Ito stochastic differential equations of a special form is presented. The explicit two-step strong numerical method of the order of accuracy 1.5 for the Ito stochastic differential equation with an additive noise is constructed in [3]. The general version of the given numerical method is obtained in [2]. The implicit two-step strong numerical methods of the orders of accuracy 1.0, 1.5 and 2.0 are presented in [2, 5], and of the orders of accuracy 2.0 and 2.5 — in [6].

Observe, that the known in the literature (see [2–6]) one-step and two-step strong numerical methods of the orders of accuracy 1.0 and 1.5 for Ito stochastic differential equations will be obtained from the represented in this work parametrical families of three-step numerical methods, if the parameters are chosen in a certain manner.

The three-step strong numerical methods of the order of accuracy 1.0

Let us consider the following three-step strong numerical methods of the order of accuracy 1.0:

$$\begin{aligned}\mathbf{y}^{(l)}_{\tau_{p+1}} &= \alpha_l \mathbf{y}^{(l)}_{\tau_p} + \beta_l \mathbf{y}^{(l)}_{\tau_{p-1}} + (1-\alpha_l-\beta_l)\mathbf{y}^{(l)}_{\tau_{p-2}} + \\ &+ \Delta[\delta_l\, \mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p+1}}, \tau_{p+1}) + \mu_l\, \mathbf{a}^{(l)}(\mathbf{y}_{\tau_p}, \tau_p) + \nu_l\, \mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-1}}, \tau_{p-1}) + \\ &+ (3-2\alpha_l-\beta_l-\delta_l-\mu_l-\nu_l)\, \mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-2}}, \tau_{p-2})] + \\ &+ (1-\alpha_l)\, \mathbf{v}^{(l)}_{\tau_p,\tau_{p-1}} + (1-\alpha_l-\beta_l)\mathbf{v}^{(l)}_{\tau_{p-1},\tau_{p-2}} + \mathbf{v}^{(l)}_{\tau_{p+1},\tau_p},\end{aligned} \tag{2}$$

where $l = 1, 2,\ldots, n$; $p = 2, 3,\ldots, N-1$; $\mathbf{g}^{(l)}$ is the l-th component of the vector $\mathbf{g}$; α_l, $\beta_l \in [0, 1]$, $\alpha_l + \beta_l \le 1$, δ_l, μ_l, ν_l are numerical parameters; $\tau_p = p\Delta$, $\Delta \in (0, 1)$; $T = N\Delta$;

$$\mathbf{v}_{\tau_{p+1},\,\tau_p} = \sum_{i=1}^{m} B_i(\mathbf{y}_{\tau_p},\tau_p)\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} + \sum_{i,j=1}^{m} G^{(j)}B_i(\mathbf{y}_{\tau_p},\tau_p)\hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p}. \tag{3}$$

Here $\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p}, \hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p}$ are approximations of the iterated Ito stochastic integrals

$$I^{(i)}_{(0)\tau_{p+1},\tau_p} = \int_{\tau_p}^{\tau_{p+1}} d\mathbf{w}^{(i)}_\tau,\ I^{(ji)}_{(00)\tau_{p+1},\tau_p} = \int_{\tau_p}^{\tau_{p+1}}\int_{\tau_p}^{s} d\mathbf{w}^{(j)}_\theta d\mathbf{w}^{(i)}_s$$

correspondingly; B_i is the i-th column of the matrix B:

$$G^{(j)} = \sum_{l=1}^{n} B^{(lj)}(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}^{(l)}},\ j = 1,\ldots, m,$$

where $B^{(lj)}$ is a component of the matrix B.

To start numerical scheme (2), (3), one needs to know the quantities $\mathbf{y}^{(l)}_{\tau_1}, \mathbf{y}^{(l)}_{\tau_2}$, $l = 1,\ldots,n$. They can be computed be means of the explicit strong one-step numerical method of the order of accuracy 1.0 [2, 3, 5, 6] (numerical method (2), (3) when $\alpha_l = \mu_l = 1$, $\beta_l = \delta_l = \nu_l = 0$). For approximation of the iterated Ito stochastic integrals, which appear in the right-hand part of (2), one may use, for example, formulas from [2, 3, 6–9].

We shall prove the convergence of numerical method (2), (3).

Theorem. Assume that $\forall$ $\mathbf{x}$, $\mathbf{y} \in \mathcal{R}^n$, s, $t \in [0, T]$, $s > t$, $p = 0, 1, \dots, N-1$, $i, j, k = 1, \dots, m$, $l = 1, \dots, n$, and for some constants K_1, K_2, K_3, C_1, C_2, which do not depend upon Δ, the following conditions:

1) $\mathbf{M}\{|\mathbf{x}_0|^2\} < \infty$;

2) $\mathbf{M}\{|\mathbf{x}_0 - \mathbf{y}_0|^2\} + \mathbf{M}\{|\mathbf{x}_{\tau_1} - \mathbf{y}_{\tau_1}|^2\} + \mathbf{M}\{|\mathbf{x}_{\tau_2} - \mathbf{y}_{\tau_2}|^2\} \le C_1\Delta^2$;

3) $|\mathbf{a}(\mathbf{x}, t) - \mathbf{a}(\mathbf{y}, t)| \le K_1|\mathbf{x} - \mathbf{y}|$, $|B_i(\mathbf{x}, t) - B_i(\mathbf{y}, t)| \le K_1|\mathbf{x} - \mathbf{y}|$,

$$|G^{(j)}B_i(\mathbf{x}, t) - G^{(j)}B_i(\mathbf{y}, t)| \le K_1|\mathbf{x} - \mathbf{y}|,$$

$$|\mathbf{a}(\mathbf{x}, t)| + |L\mathbf{a}(\mathbf{x}, t)| \le K_2(1 + |\mathbf{x}|),$$

$$|G^{(i)}\mathbf{a}(\mathbf{x}, t)| + |B_i(\mathbf{x}, t)| \le K_2(1 + |\mathbf{x}|),$$

$$|LB_i(\mathbf{x}, t)| \le K_2(1 + |\mathbf{x}|),$$

$$|G^{(j)}B_i(\mathbf{x}, t)| + |L\, G^{(j)}B_i(\mathbf{x}, t)| \le K_2(1 + |\mathbf{x}|),$$

$$|G^{(k)}G^{(j)}B_i(\mathbf{x}, t)| \le K_2(1 + |\mathbf{x}|),$$

$$|\mathbf{a}(\mathbf{x}, s) - \mathbf{a}(\mathbf{x}, t)| \le K_3(1 + |\mathbf{x}|)\sqrt{s-t},$$

$$|B_i(\mathbf{x}, s) - B_i(\mathbf{x}, t)| \le K_3(1 + |\mathbf{x}|)\sqrt{s-t},$$

$$|G^{(j)}B_i(\mathbf{x}, s) - G^{(j)}B_i(\mathbf{x}, t)| \le K_3(1 + |\mathbf{x}|)\sqrt{s-t};$$

4) $\mathbf{M}\{(I^{(i)}_{(0)\tau_{p+1},\tau_p} - \hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p})^2\} \le C_2\Delta^3$,

$\mathbf{M}\{(I^{(ji)}_{(00)\tau_{p+1},\tau_p} - \hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p})^2\} \le C_2\Delta^3$;

5) α_l, $\beta_l \in [0, 1]$, $\alpha_l + \beta_l \le 1$, $|\delta_l| + |\mu_l| + |\nu_l| < \infty$

hold true.

Then the estimation $\mathbf{M}\{|\mathbf{x}_T - \mathbf{y}_T|\} \le C\Delta$ is valid, where C is a constant, which does not depend upon Δ, $\mathbf{y}_{\tau_1}$, $\mathbf{y}_{\tau_2}$, $\mathbf{y}_T$ are determined by numerical scheme (2), (3),

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n}\mathbf{a}^{(i)}(\mathbf{x},t)\frac{\partial}{\partial \mathbf{x}^{(i)}} + \frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n}B^{(lj)}(\mathbf{x},t)B^{(ij)}(\mathbf{x},t)\frac{\partial^2}{\partial \mathbf{x}^{(l)}\,\partial \mathbf{x}^{(i)}}.$$

Proof. Using the Ito formula, we get

$$\mathbf{x}_{\tau_{p+1}} = \mathbf{x}_{\tau_p} + \Delta\mathbf{a}(\mathbf{x}_{\tau_p},\tau_p) + \mathbf{d}_{\tau_{p+1},\tau_p} + \mathbf{h}_{\tau_{p+1},\tau_p} \quad \text{a.s. (almost surely)}, \tag{4}$$

$$\mathbf{a}(\mathbf{x}_{\tau_{p+1}},\tau_{p+1}) = \mathbf{a}(\mathbf{x}_{\tau_p},\tau_p) + \mathbf{r}_{\tau_{p+1},\tau_p} \quad \text{a.s.} \tag{5}$$

Here

$$\mathbf{r}_{\tau_{p+1},\tau_p} = \int_{\tau_p}^{\tau_{p+1}} L\mathbf{a}(\mathbf{x}_s,s)ds + \sum_{i=1}^{m}\int_{\tau_p}^{\tau_{p+1}} G^{(i)}\mathbf{a}(\mathbf{x}_s,s)d\mathbf{w}_s^{(i)},$$

$$\mathbf{d}_{\tau_{p+1},\tau_p} = \sum_{i=1}^{m}B_i(\mathbf{x}_{\tau_p},\tau_p)I^{(i)}_{(0)\tau_{p+1},\tau_p} + \sum_{i,j=1}^{m}G^{(j)}B_i(\mathbf{x}_{\tau_p},\tau_p)I^{(ji)}_{(00)\tau_{p+1},\tau_p},$$

$$\mathbf{h}_{\tau_{p+1},\tau_p} = \int\limits_{\tau_p}^{\tau_{p+1}} \left(\int\limits_{\tau_p}^{s} L\mathbf{a}(\mathbf{x}_\theta,\theta)d\theta + \sum_{i=1}^{m} \int\limits_{\tau_p}^{s} G^{(i)}\mathbf{a}(\mathbf{x}_\theta,\theta)d\mathbf{w}_\theta^{(i)} \right) ds +$$

$$+ \sum_{i,j=1}^{m} \int\limits_{\tau_p}^{\tau_{p+1}} \int\limits_{\tau_p}^{s} \left(\int\limits_{\tau_p}^{\theta} LG^{(j)}B_i(\mathbf{x}_u,u)\,du + \sum_{k=1}^{m} \int\limits_{\tau_p}^{\theta} G^{(k)}G^{(j)}B_i(\mathbf{x}_u,u)d\mathbf{w}_u^{(k)} \right) \times$$

$$\times d\mathbf{w}_\theta^{(j)} d\mathbf{w}_s^{(i)} + \sum_{i=1}^{m} \int\limits_{\tau_p}^{\tau_{p+1}} \int\limits_{\tau_p}^{s} LB_i(\mathbf{x}_\theta,\theta)d\theta\, d\mathbf{w}_s^{(i)}.$$

On applying formulas (4), (5), it is not difficult to pass to the following representation:

$$\mathbf{x}_{\tau_{p+1}}^{(l)} = \alpha_l \mathbf{x}_{\tau_p}^{(l)} + \beta_l \mathbf{x}_{\tau_{p-1}}^{(l)} + (1-\alpha_l-\beta_l)\mathbf{x}_{\tau_{p-2}}^{(l)} +$$

$$+ \Delta[\delta_l\, \mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p+1}}, \tau_{p+1}) + \mu_l\, \mathbf{a}^{(l)}(\mathbf{x}_{\tau_p}, \tau_p) + \nu_l\, \mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p-1}}, \tau_{p-1}) +$$

$$+ (3 - 2\alpha_l - \beta_l - \delta_l - \mu_l - \nu_l)\, \mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p-2}}, \tau_{p-2})] +$$

$$+ (1-\alpha_l)\, \mathbf{d}_{\tau_p,\tau_{p-1}}^{(l)} + (1-\alpha_l-\beta_l)\mathbf{d}_{\tau_{p-1},\tau_{p-2}}^{(l)} + \mathbf{d}_{\tau_{p+1},\tau_p}^{(l)} +$$

$$+ Q_{\tau_{p+1},\tau_p,\tau_{p-1},\tau_{p-2}}^{(l)} \quad \text{a.s.,} \tag{6}$$

where

$$Q_{\tau_{p+1},\tau_p,\tau_{p-1},\tau_{p-2}}^{(l)} = \mathbf{h}_{\tau_{p+1},\tau_p}^{(l)} + (1-\alpha_l)\mathbf{h}_{\tau_p,\tau_{p-1}}^{(l)} + (1-\alpha_l-\beta_l)\mathbf{h}_{\tau_{p-1},\tau_{p-2}}^{(l)} -$$

$$- \Delta[\delta_l\, \mathbf{r}_{\tau_{p+1},\tau_p}^{(l)} + (\delta_l+\mu_l-1)\mathbf{r}_{\tau_p,\tau_{p-1}}^{(l)} + (\alpha_l+\delta_l+\mu_l+\nu_l-2)\,\mathbf{r}_{\tau_{p-1},\tau_{p-2}}^{(l)}];$$

the rest of notations entering (6) are the same as in (2).

Let us set

$$\mathbf{y}_s^{(l)} = \alpha_l \mathbf{y}_{\tau_p}^{(l)} + \beta_l \mathbf{y}_{\tau_{p-1}}^{(l)} + (1-\alpha_l-\beta_l)\mathbf{y}_{\tau_{p-2}}^{(l)} +$$

$$+ (s - \tau_p)\delta_l\, \mathbf{a}^{(l)}(\mathbf{y}_s, s) + \Delta[\mu_l\, \mathbf{a}^{(l)}(\mathbf{y}_{\tau_p}, \tau_p) + \nu_l\, \mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-1}}, \tau_{p-1}) +$$

$$+ (3 - 2\alpha_l - \beta_l - \delta_l - \mu_l - \nu_l)\, \mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-2}}, \tau_{p-2})] +$$

$$+ (1-\alpha_l)\mathbf{v}_{\tau_p,\tau_{p-1}}^{(l)} + (1-\alpha_l-\beta_l)\mathbf{v}_{\tau_{p-1},\tau_{p-2}}^{(l)} + \mathbf{v}_{s,\tau_p}^{(l)}. \tag{7}$$

Here $s \in (\tau_p, \tau_{p+1}]$, $p = 2, 3,\ldots, N-1$, i.e., (7) grades into (2) when $s = \tau_{p+1}$.

Subtracting (7) from (6) for $\tau_{p+1} = s$, we obtain

$$\begin{aligned} \mathbf{x}_s^{(l)} - \mathbf{y}_s^{(l)} &= \alpha_l(\mathbf{x}_{\tau_{n_s}}^{(l)} - \mathbf{y}_{\tau_{n_s}}^{(l)}) + \beta_l(\mathbf{x}_{\tau_{n_s-1}}^{(l)} - \mathbf{y}_{\tau_{n_s-1}}^{(l)}) + \\ &+ (1-\alpha_l-\beta_l)(\mathbf{x}_{\tau_{n_s-2}}^{(l)} - \mathbf{y}_{\tau_{n_s-2}}^{(l)}) + U_{s,\tau_{n_s},\tau_{n_s-1},\tau_{n_s-2}}^{(l)}, \end{aligned} \tag{8}$$

where

$$U_{s,\tau_{n_s},\tau_{n_s-1},\tau_{n_s-2}}^{(l)} = (s - \tau_p)\delta_l\, (\mathbf{a}^{(l)}(\mathbf{x}_s, s) - \mathbf{a}^{(l)}(\mathbf{y}_s, s)) +$$

$$+\Delta[\mu_l(\mathbf{a}^{(l)}(\mathbf{x}_{\tau_{n_s}},\tau_{n_s})-\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{n_s}},\tau_{n_s}))+$$

$$+\nu_l(\mathbf{a}^{(l)}(\mathbf{x}_{\tau_{n_s-1}},\tau_{n_s-1})-\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{n_s-1}},\tau_{n_s-1}))+$$

$$+(3-2\alpha_l-\beta_l-\delta_l-\mu_l-\nu_l)(\mathbf{a}^{(l)}(\mathbf{x}_{\tau_{n_s-2}},\tau_{n_s-2})-\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{n_s-2}},\tau_{n_s-2}))]+$$

$$+(1-\alpha_l)(\mathbf{d}^{(l)}_{\tau_{n_s},\tau_{n_s-1}}-\mathbf{v}^{(l)}_{\tau_{n_s},\tau_{n_s-1}})+(1-\alpha_l-\beta_l)(\mathbf{d}^{(l)}_{\tau_{n_s-1},\tau_{n_s-2}}-\mathbf{v}^{(l)}_{\tau_{n_s-1},\tau_{n_s-2}})+$$

$$+\mathbf{d}^{(l)}_{s,\tau_{n_s}}-\mathbf{v}^{(l)}_{s,\tau_{n_s}}+Q^{(l)}_{s,\tau_{n_s},\tau_{n_s-1},\tau_{n_s-2}},$$

$$n_s=\max_j\{j:\tau_j\le s\}.$$

Iterating equality (8), we obtain

$$\mathbf{x}_s^{(l)}-\mathbf{y}_s^{(l)}=e_{n_s-1}(\mathbf{x}_{\tau_2}^{(l)}-\mathbf{y}_{\tau_2}^{(l)})+f_{n_s-1}(\mathbf{x}_{\tau_1}^{(l)}-\mathbf{y}_{\tau_1}^{(l)})+g_{n_s-1}(\mathbf{x}_0^{(l)}-\mathbf{y}_0^{(l)})+$$
$$+\sum_{j=2}^{n_s-1}q_jU^{(l)}_{\tau_{n_s-j+2},\tau_{n_s-j+1},\tau_{n_s-j},\tau_{n_s-j-1}}+q_1U^{(l)}_{s,\tau_{n_s},\tau_{n_s-1},\tau_{n_s-1}}. \tag{9}$$

Here

$$e_{k+1}=\alpha_l e_k+f_k,\ e_1=\alpha_l, \tag{10}$$

$$f_{k+1}=\beta_l e_k+g_k,\ f_1=\beta_l, \tag{11}$$

$$g_{k+1}=(1-\alpha_l-\beta_l)e_k,\ g_1=1-\alpha_l-\beta_l, \tag{12}$$

$$q_{k+1}=e_k,\ q_1=1,\ k=1,2,\dots,n_s-2. \tag{13}$$

Observe, that if in the right-hand part of (2) one sets $\alpha_l=\mu_l=1$, $\beta_l=\delta_l=\nu_l=0$, then numerical method (2) will become the explicit one-step strong numerical method of the order of accuracy 1.0 known as the Milshtein method [2, 3]. The convergence of the Milshtein method is proved in [2, 3, conditions of theorem 1]. The only inherent difference, which can appear during substantiation of convergence of numerical method (2), (3) by means of approach cited in [2, 3], is that the coefficients e_k, f_k, g_k, q_k may grow when k grows, whereas for the Milshtein method $e_{n_s-1}=f_{n_s-1}=0$, $g_{n_s-1}=1$, $q_1=\dots=q_{n_s-1}=1$.

We shall show that for $\alpha_l,\beta_l\in[0,1]$, $\alpha_l+\beta_l\le 1$ the numbers e_k, f_k, g_k, q_k belong to the interval [0, 1] for all $k=1,2,\dots,n_s-1$.

It follows from (10)–(12)

$$e_{k+1}=\alpha_l e_k+\beta_l e_{k-1}+(1-\alpha_l-\beta_l)\,e_{k-2}. \tag{14}$$

Let us write down a few apparent relationships:

$$0\le e_2=(\alpha_l)^2+\beta_l\le\alpha_l+\beta_l\le 1,\ 0\le f_2=\beta_l\alpha_l+1-\alpha_l-\beta_l\le 1-\alpha_l\le 1,$$

$$e_2+f_2+g_2=1,\ 0\le g_2\le 1,\ 0\le e_3=\alpha_l e_2+f_2\le e_2+f_2=1-g_2\le 1.$$

Therefore, $e_1, e_2, e_3\in[0,1]$. By this and (14) one has

$$0\le e_4\le\alpha_l+\beta_l+1-\alpha_l-\beta_l=1.$$

Assume now that e_{k-2}, e_{k-1}, $e_k \in [0, 1]$ for some $k > 3$. The from (14) we get again

$$0 \le e_{k+1} \le \alpha_l + \beta_l + 1 - \alpha_l - \beta_l = 1,$$

i.e., by induction $e_k \in [0, 1]$ for $k = 1, 2, \ldots$.

By (12), (13) q_k, $g_k \in [0, 1]$ for $k = 1, 2, \ldots$.

It follows from (11), (12)

$$0 \le f_{k+1} = \beta_l e_k + (1 - \alpha_l - \beta_l)\, e_{k-1} \le \beta_l + 1 - \alpha_l - \beta_l = 1 - \alpha_l \le 1,$$

i.e., $f_k \in [0, 1]$ for $k = 1, 2, \ldots$.

So, it is shown that for α_l, $\beta_l \in [0, 1]$, $\alpha_l + \beta_l \le 1$ the numbers e_k, f_k, g_k, q_k belong to the interval $[0, 1]$ for $k = 1, 2, \ldots$.

It follows now from (9)

$$\mathbf{z}_s^{(l)} \stackrel{\text{def}}{=} \mathbf{M}\{|\mathbf{x}_s^{(l)} - \mathbf{y}_s^{(l)}|^2\} \le L_1\left(\sum_{k=0}^{2} \mathbf{M}\{|\mathbf{x}_{\tau_k}^{(l)} - \mathbf{y}_{\tau_k}^{(l)}|^2\} + \right.$$

$$\left. + \mathbf{M}\left\{\left(\sum_{j=2}^{n_s-1} q_j U^{(l)}_{\tau_{n_s-j+2},\tau_{n_s-j+1},\tau_{n_s-j},\tau_{n_s-j-1}}\right)^2\right\} + \mathbf{M}\{(U^{(l)}_{s,\tau_{n_s},\tau_{n_s-1},\tau_{n_s-2}})^2\}\right), \tag{15}$$

where the constant L_1 does not depend upon Δ.

Further, using conditions of the theorem as well as standard inequalities for expectations and moment properties of stochastic integrals, it is not difficult from (15) to get (see [2, 3]) the following inequality:

$$\mathbf{z}_s^{(l)} \le L_2\Delta^2 + L_3\int_0^s \mathbf{z}_u^{(l)} du, \; s \in [0, T]. \tag{16}$$

Here L_2, L_3 are constants which do not depend upon Δ.

Now by the Lyapunov and Gronwall inequality from (16) we derive

$$\mathbf{M}\{|\mathbf{x}_T^{(l)} - \mathbf{y}_T^{(l)}|\} \le \sqrt{\mathbf{z}_T^{(l)}} \le L_4\Delta,$$

i.e.,

$$\mathbf{M}\{|\mathbf{x}_T - \mathbf{y}_T|\} \le L_5\Delta,$$

where the constants L_4, L_5 do not depend upon Δ, holds true.

The theorem is proved.

Observe, that if in (2) a variable step of integration is chosen the theorem will remain valid. At that the quantity $\Delta = \max\limits_{1 \le j \le N-1} \Delta_j$, Δ_j is the j-th step of integration, will play the role of Δ.

Let us consider an Ito stochastic differential equation of the form

$$\mathrm{d}x_t = \mathrm{a}x_t\mathrm{d}t + \mathrm{b}x_t\mathrm{d}w_t. \tag{17}$$

Here a, b are constants; w_t is a standard scalar Wiener process. It is well known [2], that solution of equation (17) for $t = T$ has the form

$$x_T = x_0 e^{\left(a - \frac{1}{2}b^2\right)T + bw_T}. \tag{18}$$

Let us realize testing of numerical method (2), (3) by means of the scheme proposed in [2]. This scheme [2] is to compute the estimation $\hat{\varepsilon}_{L,M}$ of the error $\varepsilon = \mathbf{M}\{|\mathbf{x}_T - \mathbf{y}_T|\}$ by the formula

$$\hat{\varepsilon}_{L,M} = \frac{1}{LM}\sum_{k=1}^{L}\sum_{j=1}^{M}|\mathbf{x}_T^{(k,j)} - \mathbf{y}_T^{\Delta(k,j)}| \tag{19}$$

for different values of the integration step Δ; $\mathbf{y}_T^{\Delta(k,j)}$ ($k = 1,\ldots, L, j = 1,\ldots, M$) are independent realizations of the random quantity $\mathbf{y}_T = \mathbf{y}_T^{\Delta}$, which is obtained by means of the considered numerical method at the constant integration step Δ and is an approximate value of $\mathbf{x}_T$; $\mathbf{x}_T^{(k,j)}$ ($k = 1,\ldots, L$, $j = 1,\ldots, M$) are independent realizations of the random quantity $\mathbf{x}_T$ of the form (18); $\mathbf{x}_T^{(k,j)}$ and $\mathbf{y}_T^{\Delta(k,j)}$ for the fixed k and j correspond to one and the same realization of the Wiener process w_t, $t \in [0, T]$.

Numerical experiment 1 (Figure 1)

Simulate 2000 (in $M=20$ groups of $L=100$ pieces) of independent realizations of the random quantity $\mathbf{x}_T$ of the form (18) by the formula

$$x_0 e^{\left(a-\frac{1}{2}b^2\right)T+b\sqrt{\Delta}\sum_{p=1}^{N}\zeta_p} \tag{20}$$

when $x_0 = 1$, $T = 1$, $a = b = 1.5$, $N = T/\Delta$, $\Delta = 2^{-3}$, and also by means of numerical method (2), (3) for the same initial data and $y_0 = 1$, $\alpha_1 = 0$, $\beta_1 = \delta_1 = \mu_1 = \nu_1 = \frac{1}{2}$. Two first steps realize by means of the explicit Milshtein method (numerical method (2), (3) when $\alpha_1 = \mu_1 = 1$, $\beta_1 = \delta_1 = \nu_1 = 0$, Figure 1, curve 1). For simulation of the Ito stochastic integrals, which appear in (2), (3), use the formulae

$$\hat{I}^{(1)}_{(0)\tau_{p+1},\tau_p} = \sqrt{\Delta}\,\zeta_p,\quad \hat{I}^{(11)}_{(00)\tau_{p+1},\tau_p} = \frac{\Delta}{2}((\zeta_p)^2 - 1). \tag{21}$$

In formulas (20), (21) ζ_p are independent in the aggregate standard Gaussian random quantities. Compute $\hat{\varepsilon}_{100,20}$ by formula (19), repeat computations for $\Delta = 2^{-j}$, $j = 4, 5, 6, 7$, and plot the dependence $\hat{\varepsilon}_{100,20}(\Delta)$, $\Delta = 2^{-j}$, $j = 3, 4,\ldots, 7$. Repeat computations for $\alpha_1 = \beta_1 = 0$, $\delta_1 = \mu_1 = \nu_1 = \frac{3}{4}$ (Figure 1, curve 3) and $\alpha = \beta_1 = \frac{1}{3}$, $\delta_1 = \mu_1 = \nu_1 = \frac{1}{2}$ (Figure 1, curve 2).

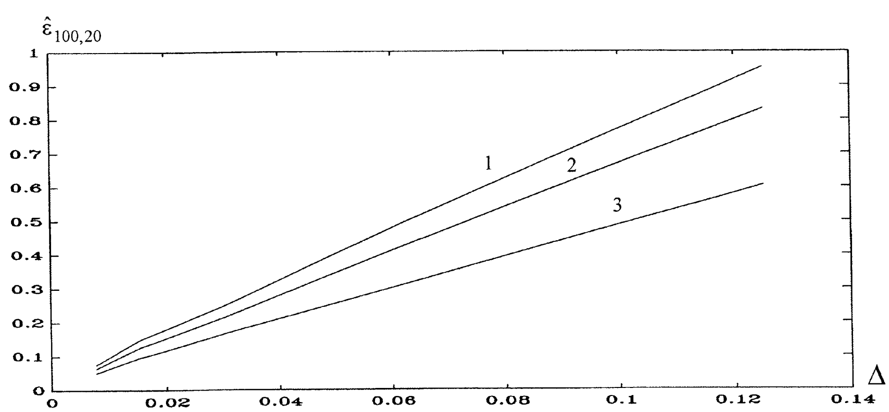

Figure 1

As is well known, presence of derivatives in expression of a numerical method is its disadvantage. In view of this we shall construct a finite-difference modification of numerical method (2), (3). For that the partial derivatives which appear in the quantities $\mathbf{v}^{(l)}_{\tau_p,\tau_{p-1}}, \mathbf{v}^{(l)}_{\tau_{p-1},\tau_{p-2}}, \mathbf{v}^{(l)}_{\tau_{p+1},\tau_p}$ in the right-hand part of (2) we shall approximate by finite-differences. For example, in (2) instead of $\mathbf{v}_{\tau_{p+1},\tau_p}$ we may use the following relationship:

$$\mathbf{v}_{\tau_{p+1},\tau_p} = \sum_{i=1}^{m} B_i(\mathbf{y}_{\tau_p},\tau_p)\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} + \\ +\frac{1}{\sqrt{\Delta}}\sum_{i,j=1}^{m}(\rho_1 B_i(\mathbf{y}_{\tau_p}+\sqrt{\Delta}\lambda_1 B_j(\mathbf{y}_{\tau_p},\tau_p),\tau_p)+ \\ +\rho_2 B_i(\mathbf{y}_{\tau_p}+\sqrt{\Delta}\lambda_2 B_j(\mathbf{y}_{\tau_p},\tau_p),\tau_p))\hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p}, \qquad (22)$$

where the constants ρ_1, ρ_2, λ_1, λ_2 satisfy the system of algebraic equations

$$\begin{cases}\rho_1+\rho_2=0,\\ \rho_1\lambda_1+\rho_2\lambda_2=1.\end{cases}$$

In particular, one may take

$$\rho_1=\frac{1}{2},\ \rho_2=-\frac{1}{2},\ \lambda_1=2,\ \lambda_2=0. \qquad (23)$$

Numerical experiment 2

Repeat numerical experiment 1 for numerical method (2), (22), (23). Two initial steps realize by means of the explicit one-step finite-difference numerical method of the order of accuracy 1.0 (numerical method (2), (22), (23) for $\alpha_1=\mu_1=1$, $\beta_1=\delta_1=\nu_1=0$).

Results of numerical experiment 2 coincide accurate up to 10^{-4} with results of numerical experiment 1.

We shall explain briefly how the convergence of numerical method (2), (22), (23) can be grounded. By means of the usual Taylor formula the right-hand part of (22) is reduced to the representation which differs from the right-hand part of (3) by a term having the order of smallness $\frac{3}{2}$ by Δ in the mean-square sense when $\Delta \downarrow 0$. Further, the convergence of numerical method (2), (22), (23) is grounded so as it has been done for numerical method (2), (3) (see the theorem).

Observe that in the right-hand part of (22) the multiplier $\frac{1}{\sqrt{\Delta}}$ occurs. However, we should not divide by the small quantity $\sqrt{\Delta}$ while realization of numerical method (2), (22), (23) since the approximation $\hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p}$ is proportional to Δ [2, 6–9] (the special case of this approximation for $i=j=1$ is given by formula (21)).

The three-step strong numerical methods of the order of accuracy 1.5

Let us develop the approach stated in the previous section. As a result we obtain the following three-step strong numerical method of the order of accuracy 1.5:

$$\mathbf{y}^{(l)}_{\tau_{p+1}} = \alpha_l\mathbf{y}^{(l)}_{\tau_p} + \beta_l\mathbf{y}^{(l)}_{\tau_{p-1}} + (1-\alpha_l-\beta_l)\mathbf{y}^{(l)}_{\tau_{p-2}} +$$

$$+\Delta[\delta_l\,\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p+1}},\tau_{p+1}) + \mu_l\,\mathbf{a}^{(l)}(\mathbf{y}_{\tau_p},\tau_p) + \nu_l\,\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-1}},\tau_{p-1}) +$$

$$+ (3 - 2\alpha_l - \beta_l - \delta_l - \mu_l - \nu_l)\, \mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-2}}, \tau_{p-2})] +$$

$$+ \Delta^2\, [\varepsilon_l\, L\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p+1}}, \tau_{p+1}) + \lambda_l\, L\mathbf{a}^{(l)}(\mathbf{y}_{\tau_p}, \tau_p) + \varphi_l L\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-1}}, \tau_{p-1}) +$$

$$+\left(\frac{9}{2} - 2\alpha_l - \frac{1}{2}\beta_l - 3\delta_l - 2\mu_l - \nu_l - \varepsilon_l - \lambda_l - \varphi_l\right) L\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-2}}, \tau_{p-2})\Big] -$$

$$-\Delta \sum_{i=1}^{m} [\delta_l G^{(i)}\mathbf{a}^{(l)}(\mathbf{y}_{\tau_p}, \tau_p)\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} + (\delta_l + \mu_l - 1) G^{(i)}\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-1}}, \tau_{p-1}) \times \tag{24}$$

$$\times \hat{I}^{(i)}_{(0)\tau_p,\tau_{p-1}} + (\alpha_l + \delta_l + \mu_l + \nu_l - 2) G^{(i)}\mathbf{a}^{(l)}(\mathbf{y}_{\tau_{p-2}}, \tau_{p-2})\hat{I}^{(i)}_{(0)\tau_{p-1},\tau_{p-2}}] +$$

$$+ (1 - \alpha_l)\, \mathbf{v}^{(l)}_{\tau_p,\tau_{p-1}} + (1 - \alpha_l - \beta_l)\mathbf{v}^{(l)}_{\tau_{p-1},\tau_{p-2}} + \mathbf{v}^{(l)}_{\tau_{p+1},\tau_p}.$$

Here

$$\mathbf{v}_{\tau_{p+1},\tau_p} = \sum_{i=1}^{m} B_i(\mathbf{y}_{\tau_p}, \tau_p)\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} + \sum_{i,j=1}^{m} G^{(j)} B_i(\mathbf{y}_{\tau_p}, \tau_p)\hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p} +$$
$$+ \sum_{i=1}^{m} [G^{(j)}\mathbf{a}(\mathbf{y}_{\tau_p}, \tau_p)(\Delta\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} + \hat{I}^{(i)}_{(1)\tau_{p+1},\tau_p}) - LB_i(\mathbf{y}_{\tau_p}, \tau_p)\hat{I}^{(i)}_{(1)\tau_{p+1},\tau_p}] + \tag{25}$$
$$+ \sum_{i,j,k=1}^{m} G^{(k)} G^{(j)} B_i(\mathbf{y}_{\tau_p}, \tau_p)\hat{I}^{(kji)}_{(000)\tau_{p+1},\tau_p},$$

$\hat{I}^{(i)}_{(1)\tau_{p+1},\tau_p}, \hat{I}^{(kji)}_{(000)\tau_{p+1},\tau_p}$ are approximations of the iterated Ito stochastic integrals

$$I^{(i)}_{(1)\tau_{p+1},\tau_p} = \int_{\tau_p}^{\tau_{p+1}} (\tau_p - s) d\mathbf{w}^{(i)}_s,\; I^{(kji)}_{(000)\tau_{p+1},\tau_p} = \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{s} \int_{\tau_p}^{\theta} d\mathbf{w}^{(k)}_u d\mathbf{w}^{(j)}_\theta d\mathbf{w}^{(i)}_s;$$

α_l, $\beta_l \in [0, 1]$; $\alpha_l + \beta_l \le 1$; δ_l, μ_l, ν_l, ε_l, λ_l, φ_l are numerical parameters; the rest notations occurring in (24) are the same as in (2).

Observe, that when $\varepsilon_l = \lambda_l = \varphi_l = 0$, $\frac{9}{2} - 2\alpha_l - \frac{1}{2}\beta_l - 3\delta_l - 2\mu_l - \nu_l - \varepsilon_l - \lambda_l - \varphi_l = 0$ the less quantity of partial derivatives will enter the right-hand part of (24) than the explicit one-step numerical method of the order of accuracy 1.5 (numerical method (24) for $\beta_l = 0$, $\alpha_l = 1$, $\delta_l = 0$, $\mu_l = 1$, $\nu_l = 0$, $\varepsilon_l = 0$, $\lambda_l = \frac{1}{2}$, $\varphi_l = 0$) since the term $\frac{\Delta^2}{2} L\mathbf{a}(\mathbf{y}_{\tau_p}, \tau_p)$ enters the right-hand part of the latter one) [2, 3].

Without proof of the convergence of numerical method (24), (25) we shall note that it can be carried out similarly to a proof of the theorem with the use of the representation

$$\mathbf{x}^{(l)}_{\tau_{p+1}} = \alpha_l \mathbf{x}^{(l)}_{\tau_p} + \beta_l \mathbf{y}^{(l)}_{\tau_{p-1}} + (1 - \alpha_l - \beta_l)\mathbf{x}^{(l)}_{\tau_{p-2}} +$$

$$+ \Delta[\delta_l\, \mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p+1}}, \tau_{p+1}) + \mu_l\, \mathbf{a}^{(l)}(\mathbf{x}_{\tau_p}, \tau_p) + \nu_l\, \mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p-1}}, \tau_{p-1}) +$$

$$+ (3 - 2\alpha_l - \beta_l - \delta_l - \mu_l - \nu_l)\, \mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p-2}}, \tau_{p-2})] +$$

$$+ \Delta^2\, [\varepsilon_l\, L\mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p+1}}, \tau_{p+1}) + \lambda_l\, L\mathbf{a}^{(l)}(\mathbf{x}_{\tau_p}, \tau_p) + \varphi_l L\mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p-1}}, \tau_{p-1}) + \tag{26}$$

$$+\left(\frac{9}{2} - 2\alpha_l - \frac{1}{2}\beta_l - 3\delta_l - 2\mu_l - \nu_l - \varepsilon_l - \lambda_l - \varphi_l\right) L\mathbf{a}^{(l)}(\mathbf{x}_{\tau_{p-2}}, \tau_{p-2})\Big] -$$

$$-\Delta[\delta_l \mathbf{q}^{(l)}_{\tau_{p+1},\tau_p} + (\delta_l + \mu_l - 1)\mathbf{q}^{(l)}_{\tau_p,\tau_{p-1}} + (\alpha_l + \delta_l + \mu_l + \nu_l - 2)\mathbf{q}^{(l)}_{\tau_{p-1},\tau_{p-2}}] +$$

$$+ (1-\alpha_l)\ \mathbf{g}^{(l)}_{\tau_p,\tau_{p-1}} + (1-\alpha_l-\beta_l)\mathbf{g}^{(l)}_{\tau_{p-1},\tau_{p-2}} + \mathbf{g}^{(l)}_{\tau_{p+1},\tau_p} \ \text{a.s.},$$

where

$$\mathbf{q}_{\tau_{p+1},\tau_p} = \mathbf{r}_{\tau_{p+1},\tau_p} + \sum_{i=1}^{m} G^{(i)}\mathbf{a}(\mathbf{x}_{\tau_p},\tau_p) I^{(i)}_{(0)\tau_{p+1},\tau_p},$$

$$\mathbf{g}_{\tau_{p+1},\tau_p} = \mathbf{d}_{\tau_{p+1},\tau_p} + \mathbf{h}_{\tau_{p+1},\tau_p},$$

$$\mathbf{d}_{\tau_{p+1},\tau_p} = \sum_{i=1}^{m} B_i(\mathbf{x}_{\tau_p},\tau_p) I^{(i)}_{(0)\tau_{p+1},\tau_p} + \sum_{i,j=1}^{m} G^{(j)} B_i(\mathbf{x}_{\tau_p},\tau_p) I^{(ji)}_{(00)\tau_{p+1},\tau_p} +$$
$$+ \sum_{i=1}^{m} [G^{(i)}\mathbf{a}(\mathbf{x}_{\tau_p},\tau_p)(\Delta I^{(i)}_{(0)\tau_{p+1},\tau_p} + I^{(i)}_{(1)\tau_{p+1},\tau_p}) - LB_i(\mathbf{x}_{\tau_p},\tau_p) I^{(i)}_{(1)\tau_{p+1},\tau_p}] +$$
$$+ \sum_{i,j,k=1}^{m} G^{(k)} G^{(j)} B_i(\mathbf{x}_{\tau_p},\tau_p) I^{(kji)}_{(000)\tau_{p+1},\tau_p},$$

$$\mathbf{r}_{\tau_{p+1},\tau_p} = \int_{\tau_p}^{\tau_{p+1}} \left(\int_{\tau_p}^{s} LL\mathbf{a}(\mathbf{x}_\theta,\theta)d\theta + \sum_{i=1}^{m} \int_{\tau_p}^{s} G^{(i)} L\mathbf{a}(\mathbf{x}_\theta,\theta) d\mathbf{w}^{(i)}_\theta \right) ds +$$

$$+ \sum_{i=1}^{m} \int_{\tau_p}^{\tau_{p+1}} \left(\int_{\tau_p}^{s} LG^{(i)}\mathbf{a}(\mathbf{x}_\theta,\theta)d\theta + \sum_{j=1}^{m} \int_{\tau_p}^{s} G^{(j)} G^{(i)}\mathbf{a}(\mathbf{x}_\theta,\theta) d\mathbf{w}^{(j)}_\theta \right) d\mathbf{w}^{(i)}_s,$$

$$\mathbf{h}_{\tau_{p+1},\tau_p} = \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{s} \left(\int_{\tau_p}^{\theta} LL\mathbf{a}(\mathbf{x}_u,u)du + \sum_{i=1}^{m} \int_{\tau_p}^{\theta} G^{(i)} L\mathbf{a}(\mathbf{x}_u,u) d\mathbf{w}^{(i)}_u \right) d\theta ds +$$

$$+ \sum_{i=1}^{m} \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{s} \left(\int_{\tau_p}^{\theta} LG^{(i)}\mathbf{a}(\mathbf{x}_u,u)du + \sum_{j=1}^{m} d\mathbf{w}^{(j)}_u \right) d\mathbf{w}^{(i)}_\theta ds +$$

$$+ \sum_{i=1}^{m} \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{s} \left(\int_{\tau_p}^{\theta} LLB_i(\mathbf{x}_u,u)du + \sum_{j=1}^{m} \int_{\tau_p}^{\theta} G^{(j)} LB_i(\mathbf{x}_u,u) d\mathbf{w}^{(j)}_u \right) d\theta d\mathbf{w}^{(i)}_s +$$

$$+ \sum_{i,j=1}^{m} \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{s} \left(\int_{\tau_p}^{\theta} LG^{(j)} B_i(\mathbf{x}_u,u)du + \sum_{k=1}^{m} \int_{\tau_p}^{\theta} G^{(k)} G^{(j)} B_i(\mathbf{x}_u,u) d\mathbf{w}^{(k)}_u \right) d\mathbf{w}^{(j)}_\theta d\mathbf{w}^{(i)}_s;$$

the rest notations entering (26) are the same as in (24), (25).

Note that representation (26) is obtained with the use of the relationships

$$\mathbf{x}_{\tau_{p+1}} = \mathbf{x}_{\tau_p} + \Delta\mathbf{a}(\mathbf{x}_{\tau_p},\tau_p) + \frac{\Delta^2}{2} L\mathbf{a}(\mathbf{x}_{\tau_p},\tau_p) + \mathbf{d}_{\tau_{p+1},\tau_p} + \mathbf{h}_{\tau_{p+1},\tau_p} \ \text{a.s.},$$

$$\mathbf{a}(\mathbf{x}_{\tau_{p+1}},\tau_{p+1}) = \mathbf{a}(\mathbf{x}_{\tau_p},\tau_p) + \Delta L\mathbf{a}(\mathbf{x}_{\tau_p},\tau_p) + \sum_{i=1}^{m} G^{(i)}\mathbf{a}(\mathbf{x}_{\tau_p},\tau_p) I^{(i)}_{(0)\tau_{p+1},\tau_p} + \mathbf{r}_{\tau_{p+1},\tau_p} \ \text{a.s.},$$

which, in their turn, can be obtained by the Ito formula.

Numerical experiment 3 (Figure 2)

Repeat numerical experiment 1 for numerical method (24), (25) for the following ways of choice of parameters:

1) $\alpha_1 = \beta_1 = 0,\ \varepsilon_1 = \varphi_1 = \lambda_1 = 0,\ \mu_1 = \nu_1 = 1,\ \delta_1 = \frac{1}{2}$ (Figure 2, curve 3);

2) $\alpha_1 = \beta_1 = 0,\ \varepsilon_1 = \varphi_1 = \lambda_1 = 0,\ \mu_1 = \nu_1 = \frac{1}{2},\ \delta_1 = 1$ (Figure 2, curve 1);

3) $\alpha_1 = 0,\ \beta_1 = \frac{1}{2},\ \varepsilon_1 = \varphi_1 = \lambda_1 = 0,\ \mu_1 = \frac{1}{2},\ \nu_1 = 1,\ \delta_1 = \frac{3}{4}$ (Figure 2, curve 2).

For simulation of the iterated Ito stochastic integral $I^{(111)}_{(000)\tau_{p+1},\tau_p}$ use the formula

$$\hat{I}^{(111)}_{(000)\tau_{p+1},\tau_p} = \frac{\Delta^{3/2}}{6}\left((\zeta_p)^3 - 3\zeta_p\right),$$

where ζ_p are the same random quantities as in numerical experiment 1. Do not simulate the Ito stochastic integral $I^{(1)}_{(1)\tau_{p+1},\tau_p}$, since its coefficient will turn out to be zero in the expression which is obtained after application of numerical method (24), (25) to equation (17).

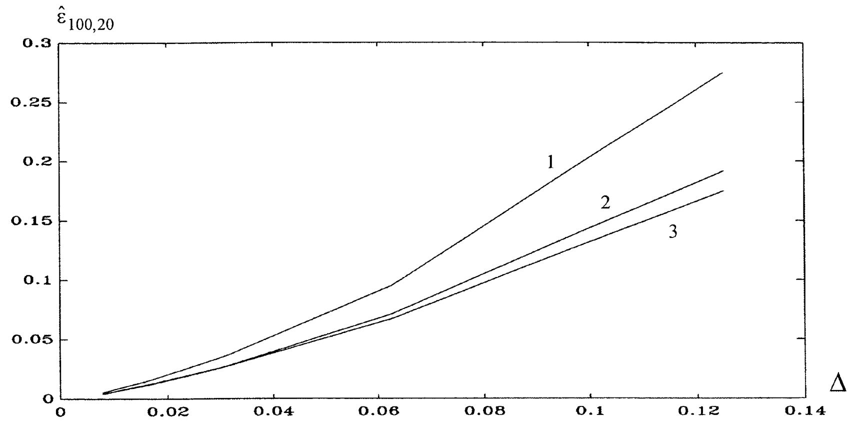

Figure 2

We shall construct a finite-difference modification of numerical method (24), (25). For that we shall approximate by finite-differences the partial derivatives which appear in the right-hand part of (25). As a result, in (24), instead of $\mathbf{v}_{\tau_{p+1},\tau_p}$ of the form (25), we shall use the relationship

$$\mathbf{v}_{\tau_{p+1},\tau_p} = \sum_{i=1}^{m} B_i(\mathbf{y}_{\tau_p},\tau_p)\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} +$$

$$+\frac{1}{\sqrt{\Delta}}\sum_{i,j=1}^{m}\sum_{k=1}^{2}\theta_k B_i(\mathbf{y}_{\tau_p} + \sqrt{\Delta}\,\eta_k B_j(\mathbf{y}_{\tau_p},\tau_p),\tau_p)\hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p} +$$

$$+\frac{1}{\Delta}\sum_{i=1}^{m}\left[\sum_{k=1}^{2}\pi_k \mathbf{a}(\mathbf{y}_{\tau_p} + \Delta\sigma_k B_i(\mathbf{y}_{\tau_p},\tau_p),\tau_p)(\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p} + \hat{I}^{(i)}_{(1)\tau_{p+1},\tau_p}) -\right.$$

$$-\left(\frac{1}{2}\sum_{r=1}^{m}\sum_{k=1}^{3}\rho_k B_i(\mathbf{y}_{\tau_p} + \sqrt{\Delta}\,\omega_k B_r(\mathbf{y}_{\tau_p},\tau_p),\tau_p) + \sum_{k=1}^{2}\chi_k B_i(\mathbf{y}_{\tau_p} + \Delta\psi_k \mathbf{a}(\mathbf{y}_{\tau_p},\tau_p),\tau_p + \gamma_k\Delta)\right)\times$$

$$\times \hat{I}^{(i)}_{(1)\tau_{p+1},\tau_p}\Bigg] + \frac{1}{\Delta}\sum_{i,j,q=1}^{m}\sum_{k=1}^{2}\theta_k \mathbf{k}_{ij}(\mathbf{y}_{\tau_p} + \sqrt{\Delta}\,\eta_k B_q(\mathbf{y}_{\tau_p},\tau_p),\tau_p)\,\hat{I}^{(qji)}_{(000)\tau_{p+1},\tau_p}. \tag{27}$$

Here

$$\mathbf{k}_{ij}(\mathbf{y},s) = \sum_{k=1}^{2}\pi_k B_i(\mathbf{y} + \sqrt{\Delta}\,\sigma_k B_j(\mathbf{y},s),s),$$

$\mathbf{y} \in \mathscr{R}^n$, $s \in [0, T]$; the parameters θ_k, η_k, π_k, σ_k, χ_k, ψ_k, γ_k, ρ_i, ω_i, $k = 1, 2$, $i = 1, 2, 3$, satisfy the following systems of algebraic equations:

$$\begin{cases}\sum_{i=1}^{2}\theta_i = 0,\\ \sum_{i=1}^{2}\theta_i\eta_i = 1,\\ \sum_{i=1}^{2}\theta_i\eta_i^2 = 0,\end{cases}\qquad \begin{cases}\sum_{i=1}^{2}\pi_i = 0,\\ \sum_{i=1}^{2}\pi_i\sigma_i = 1,\end{cases}\qquad \begin{cases}\sum_{i=1}^{3}\rho_i = 0,\\ \sum_{i=1}^{3}\rho_i\omega_i = 0,\\ \sum_{i=1}^{3}\rho_i\frac{\omega_i^2}{2} = 1,\end{cases}\qquad \begin{cases}\sum_{i=1}^{2}\chi_i = 0,\\ \sum_{i=1}^{2}\chi_i\gamma_i = 1,\\ \sum_{i=1}^{2}\chi_i\psi_i = 1.\end{cases}$$

For example, we may take

$$\begin{gathered}\theta_1 = \rho_1 = \rho_2 = \omega_3 = \chi_1 = 1,\ \theta_2 = \chi_2 = -1,\ \pi_1 = \eta_1 = \gamma_1 = \psi_1 = \frac{1}{2},\\ \pi_2 = \eta_2 = \gamma_2 = \psi_2 = -\frac{1}{2},\ \ \sigma_2 = \omega_1 = 0,\ \sigma_1 = \omega_2 = 2,\ \rho_3 = -2.\end{gathered} \tag{28}$$

Observe, that the convergence of numerical method (24), (27), (28) may be grounded just as the convergence of numerical method (2), (22). In the right-hand part of (27) the multipliers $\frac{1}{\sqrt{\Delta}}, \frac{1}{\Delta}$ occur. However, we should not divide by the small quantities $\sqrt{\Delta}, \Delta$ while realization of numerical method (24), (27), (28), since the approximations $\hat{I}^{(i)}_{(0)\tau_{p+1},\tau_p}$, $\hat{I}^{(i)}_{(1)\tau_{p+1},\tau_p}$, $\hat{I}^{(ji)}_{(00)\tau_{p+1},\tau_p}$, $\hat{I}^{(qji)}_{(000)\tau_{p+1},\tau_p}$ are proportional correspondingly to the quantities $\sqrt{\Delta}$, $\Delta\sqrt{\Delta}$, Δ and $\Delta\sqrt{\Delta}$ [2, 6–9].

Numerical experiment 4 (Figure 3)

Repeat numerical experiment 3 for numerical method (24), (27), (28). For simulation of the stochastic integral $I^{(1)}_{(1)\tau_{p+1},\tau_p}$ use the formula

$$\hat{I}^{(1)}_{(1)\tau_{p+1},\tau_p} = -\frac{\Delta^{3/2}}{2}\left(\zeta_p + \frac{1}{\sqrt{3}}\xi_p\right),$$

where ζ_p, ξ_p, $p = 0, 1,\ldots, N-1$, are independent in the aggregate standard Gaussian random quantities, and ζ_p are the same random quantities as in numerical experiment 1. Two initial steps realize by means of the explicit one-step strong finite-difference numerical method of the order of accuracy 1.5 of the following form [6]:

$$\begin{aligned}\mathbf{y}_{\tau_{p+1}} = \mathbf{y}_{\tau_p} + \frac{\Delta}{2}\Bigg(\frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{3}\mu_j\,\mathbf{a}(\mathbf{y}_{\tau_p} + \sqrt{\Delta}\,\lambda_j B_r(\mathbf{y}_{\tau_p},\tau_p),\tau_p) + \\ + \sum_{j=1}^{2}\rho_j\,\mathbf{a}(\mathbf{y}_{\tau_p} + \Delta\nu_j\,\mathbf{a}(\mathbf{y}_{\tau_p},\tau_p),\tau_p + \varphi_j\Delta)\Bigg) + \mathbf{v}_{\tau_{p+1},\tau_p}.\end{aligned}$$

Here $\mathbf{v}_{\tau_{p+1},\tau_p}$ has the form (27), (28); $\rho_1 = \rho_2 = 1$, $\nu_1 = 0$, $\nu_2 = 1$, $\varphi_1 = 0$, $\varphi_2 = 1$, $\mu_1 = \mu_2 = 1$, $\mu_3 = -2$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 0$.

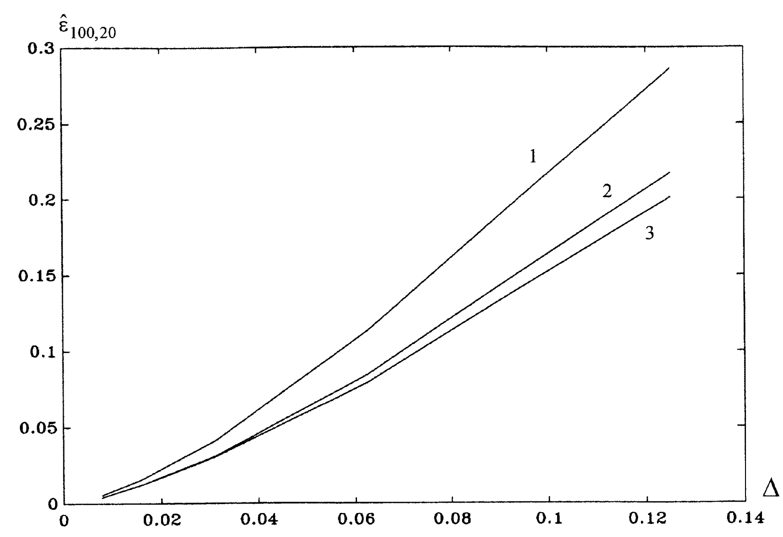

Figure 3

While comparing figures 2 and 3, we notice that errors yielded by numerical method (24), (27), (28), which is as it should be, are lightly larger than the corresponding errors yielded by numerical method (24), (25).

Observe, that in [2, 3, 5, 6] two-step strong numerical methods for Ito stochastic differential equations in which parameters belonged to the interval [0, 1] were considered. We shall give an example of three-step strong numerical methods of the order of accuracy 1.5 in which not all parameters belong to the interval [0, 1].

Numerical experiment 5 (Figure 4)

Repeat numerical experiment 3 for numerical method (24), (25) for $\alpha_1 = \beta_1 = 0$, $\delta_1 = 5$, $\mu_1 = -8$, $\nu_1 = \frac{11}{2}$, $\varepsilon_1 = \varphi_1 = \lambda_1 = 0$, $\Delta = 2^{-j}$, $j = 5, 6, \ldots, 9$.

In this work we show that one of sufficient conditions which ensures the convergence of three-step strong numerical methods for Ito stochastic differential equations is the following condition: $\alpha_l, \beta_l \in [0, 1]$, $\alpha_l + \beta_l \leq 1$. In conclusion we shall cite the numerical example in which non-fulfillment of this conditions results in breaking the convergence of the considered three-step numerical method.

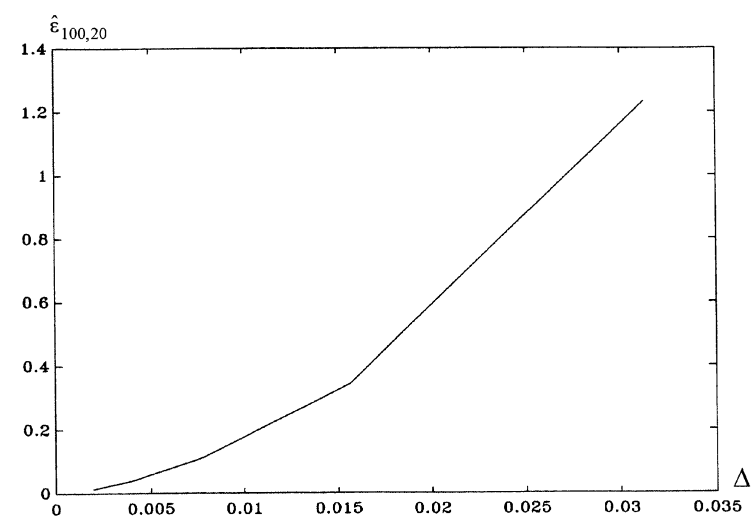

Figure 4

Numerical experiment 6 (Figure 5)

Repeat numerical experiment 3 for numerical method (24), (25) for $\alpha_1 = 0$, $\beta_1 = 2$, $\delta_1 = \frac{1}{2}$, $\mu_1 = \frac{1}{2}$, $\nu_1 = 1$, $\varepsilon_1 = \varphi_1 = \lambda_1 = 0$, $\Delta = 2^{-j}$, $j = 1, 2, 3, 4$.

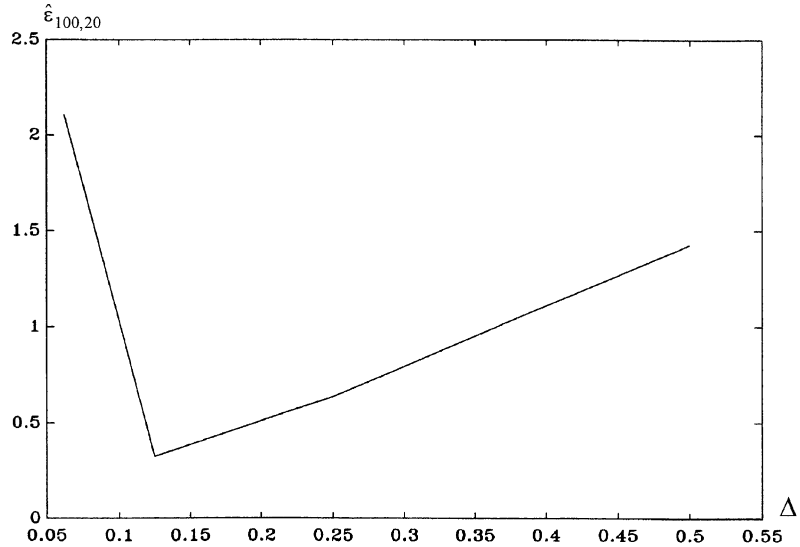

Figure 5

References

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