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Combined Method of Strong Approximation of Multiple Stochastic Integrals

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A method of mean square approximation of multiple stochastic integrals, which is a combination of methods based on multiple Fourier series and multiple integral sums, is developed. The mean square convergence of the method is investigated. The method with $N = 2$ is more economic, than the method of multiple Fourier series.

Key words: stochastic differential equation, stochastic integral, mean square approximation, Fourier series, convergence, computation accuracy.

Let a probabilistic space $(\Omega, \mathcal{F}, \mathrm{P})$ and a nondecreasing, continuous from the right aggregate $\{\mathcal{F}_t, t \in [0, T]\}$ of σ-subalgebras $\mathcal{F}$ be prescribed. Consider an Ito stochastic differential equation in integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_s, s)ds + \int\limits_0^t B(\mathbf{x}_s, s)d\mathbf{w}_s, \tag{1}$$

where $t \in [0, T]$; $\mathbf{x}_s \in \Re^n$ is a solution of equation (1), $\mathbf{w}_t$ is an m-dimensional and $\mathcal{F}_t$-measurable for all $t \in [0, T]$ standard Wiener process with independent components $\mathbf{w}_s^{(i)}$ $(i = 1,\ldots, m)$; $\mathbf{x}_0 \in \Re^n$ is the initial condition which is stochastically independent of increments $\mathbf{w}_t - \mathbf{w}_0$ for $t > 0$ and $\mathrm{M}\{|x_0|^2\} < \infty$; non-random functions $\mathbf{a}(\mathbf{x}, t) : \Re^n \times [0, T] \to \Re^n$, $B(\mathbf{x}, t) : \Re^n \times [0, T] \to \Re^{n\times m}$ satisfy conditions of existence and uniqueness of solution of equation (1) in the sense of stochastic equivalence [1].

Numerical integration of Ito stochastic differential equations is a very actual problem because these equations are adequate mathematical models of a number of physical and technical systems [2, 3]. They are also used in solutions of mathematical problems [2, 4]. The present paper is devoted to some computational issues of this problem associated with numerical modeling of multiple stochastic integrals.

It is known, that one of stages in construction of strong [2] numerical methods for equation (1) consists in approximation at each integration step of systems of multiple stochastic integrals of the form

$$I[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{2}$$

or

$$I^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

taking into account the following criteria:

$$\mathrm{M}\{(I[\psi^{(k)}]_{T,t} - \hat{I}[\psi^{(k)}]_{T,t})^2\} \le \varepsilon,$$

$$\mathrm{M}\{(I^*[\psi^{(k)}]_{T,t} - \hat{I}^*[\psi^{(k)}]_{T,t})^2\} \le \varepsilon.$$

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Here $\int, \overset{*}{\int}$ are respectively Ito and Stratonovich stochastic integrals; $i_1, \ldots, i_k = 0, 1, \ldots, m$; $\mathbf{w}_t^{(0)} \overset{\text{def}}{=} t$; $\hat{I}[\psi^{(k)}]_{T,t}$, $\hat{I}^*[\psi^{(k)}]_{T,t}$ are approximations of integrals $I[\psi^{(k)}]_{T,t}, I^*[\psi^{(k)}]_{T,t}$; ε is the accuracy of approximation. At that in [2, 4] systems (2), (3) were taken for $\psi_1(s) = \ldots = \psi_k(s) \equiv 1, i_1, \ldots, i_k = 0, 1, \ldots, m$, and in [3] for $\psi_j(s) = (t-s)^{l_j}$ or $\psi_j(s) = (T-s)^{l_j}, l_j = 0, 1, 2, \ldots, j = 1, \ldots, k$, $i_1, \ldots, i_k = 1, 2, \ldots, m$.

In [5] the method of multiple Fourier series of approximation of integrals (3) and the method of multiple integral sums of approximation of integrals (2) were considered. The author of [5] noted, that the first method converges in mean-square sense several times faster, than the second one (we mean computer computation time necessary for attaining prescribed approximation accuracy).

However, in realization of the method of multiple Fourier series one should do a lot of additional work for calculation of coefficients of multiple Fourier series. This work is rather time-consuming, since analytical formulae for necessary Fourier coefficients were obtained in [2–4, 6] only for multiplicity m of Fourier series, equal to 1–3. If $m > 3$, then each coefficient is to be calculated separately [6] using software like DERIVE or MAPLE. Besides, even in the case $m = 3$ these coefficients are calculated by rather complicated formulae [6, Appendix]. Note further, that under insignificant increasing of accuracy of approximation of the considered multiple stochastic integral, the size of coefficient array for the corresponding multiple Fourier series significantly increases [6].

In this paper we suggest a combined method of approximation of integrals (2). It unites methods, based on multiple Fourier series and multiple integral sums. The method also enables one to reduce significantly the total number of coefficients of multiple Fourier series necessary for approximation of the studied multiple stochastic integral. (At that, the approximation accuracy remains the same and computational efforts are about the same, as in the method of multiple Fourier series.)

Formulae for approximations of multiple stochastic integrals by combined method

In this section we consider formulae for approximation of integrals (2), which are necessary for realization of a strong numerical method of accuracy order 1.5 for equation (1) [2].

Using the additivity property of Ito stochastic integral, we can write

$$I_{0_{T,t}}^{(i_1)} = \sqrt{\Delta} \sum_{k=0}^{N-1} \zeta_{0,k}^{(i_1)} \quad \text{a.s.,} \tag{4}$$

$$I_{1_{T,t}}^{(i_1)} = \sum_{k=0}^{N-1} \left(I_{1_{\tau_{k+1},\tau_k}}^{(i_1)} - \Delta^{3/2} k \zeta_{0,k}^{(i_1)} \right) \quad \text{a.s.,} \tag{5}$$

$$I_{00_{T,t}}^{(i_1 i_2)} = \Delta \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \zeta_{0,k}^{(i_2)} \zeta_{0,l}^{(i_1)} + \sum_{k=0}^{N-1} I_{00_{\tau_{k+1},\tau_k}}^{(i_1 i_2)} \quad \text{a.s.,} \tag{6}$$

$$I_{000_{T,t}}^{(i_1 i_2 i_3)} = \Delta^{3/2} \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \sum_{q=0}^{l-1} \zeta_{0,k}^{(i_3)} \zeta_{0,l}^{(i_2)} \zeta_{0,q}^{(i_1)} +$$

$$+ \sqrt{\Delta} \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \left(\zeta_{0,k}^{(i_3)} I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} + \zeta_{0,l}^{(i_1)} I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)} \right) + \sum_{k=0}^{N-1} I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)} \quad \text{a.s.,} \tag{7}$$

where

$$I_{l_1 \ldots l_{k_{T,t}}}^{(i_1 \ldots i_k)} = \int_t^T (t - t_k)^{l_k} \ldots \int_t^{t_2} (t - t_1)^{l_1} \, d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)};$$

$l_1,\ldots,l_k=0,1,\ldots;\ \ i_1,\ldots,i_k=1,\ldots,m;\ \ T-t=n\Delta;\ \ \tau_k=t+k\Delta;\ \ \zeta_{0,k}^{(i)}\stackrel{\rm def}{=}\Delta^{-1/2}\int\limits_{\tau_k}^{\tau_{k+1}}d\mathbf{w}_s^{(i)};\ \ k=0,1,\ldots,N-1;$ $N<\infty$; the sum over an empty set is regarded as zero.

Substituting into (5) of the relationship [6]

$$I_{1_{\tau_{k+1},\tau_k}}^{(i_1)}=-\frac{\Delta^{3/2}}{2}\left(\zeta_{0,k}^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_{1,k}^{(i_1)}\right)\ \text{a.s.},$$

here $\zeta_{0,k}^{(i_1)},\ \zeta_{1,k}^{(i_1)}$ are independent standard Gaussian random variables, we obtain

$$I_{1_{T,t}}^{(i_1)}=-\Delta^{3/2}\sum_{k=0}^{N-1}\left(\left(\frac{1}{2}+k\right)\zeta_{0,k}^{(i_1)}+\frac{1}{2\sqrt{3}}\zeta_{1,k}^{(i_1)}\right)\ \text{a.s.} \tag{8}$$

Approximate integrals $I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)},\ I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)},\ I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)}$, that enter the right-hand sides of (6), (7), by the method of multiple Fourier series in Legendre polynomials [6]. In the result we obtain

$$I_{00_{T,t}}^{(i_1i_2)N,q}=\Delta\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\zeta_{0,k}^{(i_2)}\zeta_{0,l}^{(i_1)}+\sum_{k=0}^{N-1}I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)q}, \tag{9}$$

$$I_{000_{T,t}}^{(i_1i_2i_3)N,q_1,q_2}=\Delta\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\sum_{q=0}^{l-1}\zeta_{0,k}^{(i_3)}\zeta_{0,l}^{(i_2)}\zeta_{0,q}^{(i_1)}+$$

$$+\sqrt{\Delta}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\left(\zeta_{0,k}^{(i_3)}I_{00_{\tau_{l+1},\tau_l}}^{(i_1i_2)q_1}+\zeta_{0,l}^{(i_1)}I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)q_1}\right)+\sum_{k=0}^{N-1}I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)q_2}, \tag{10}$$

where

$$I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)q}=\frac{\Delta}{2}\left(\zeta_{0,k}^{(i_1)}\zeta_{0,k}^{(i_2)}+\sum_{l=1}^{q}\frac{1}{\sqrt{4l^2-1}}\left(\zeta_{l-1,k}^{(i_1)}\zeta_{l,k}^{(i_2)}-\zeta_{l,k}^{(i_1)}\zeta_{l-1,k}^{(i_2)}\right)-\mathbf{1}_{\{i_1=i_2\}}\right), \tag{11}$$

$$I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)q_2}=\sum_{i,j,l=0}^{q_2}C_{ijl}\zeta_{l,k}^{(i_1)}\zeta_{j,k}^{(i_2)}\zeta_{i,k}^{(i_3)}+\frac{1}{2}\mathbf{1}_{\{i_3=i_2\}}I_{1_{\tau_{k+1},\tau_k}}^{(i_1)}-\frac{1}{2}\mathbf{1}_{\{i_2=i_1\}}\left(\Delta^{3/2}\zeta_{0,k}^{(i_3)}+I_{1_{\tau_{k+1},\tau_k}}^{(i_1)}\right), \tag{12}$$

$$C_{ijl}=\frac{\sqrt{(2i+1)(2j+1)(2l+1)}}{8}\Delta^{3/2}\bar{C}_{ijl}, \tag{13}$$

$$\bar{C}_{ijl}=\int\limits_{-1}^{1}\int\limits_{-1}^{z}\int\limits_{-1}^{y}P_l(x)dx\,P_j(y)dy\,P_i(z)dz. \tag{14}$$

Here $P_i(x)$ is the Legendre polynomial, $\mathbf{1}_A$ is the indicator of set A, $\zeta_{l,k}^{(i)}$ are independent for different l, k or i standard Gaussian random variables; $\Delta=(T-t)/N$; the sum over an empty set is regarded as zero.

In particular, for $N=2$ formulae (4), (8)–(10) take the following form:

$$I_{0_{T,t}}^{(i_1)}=\sqrt{\Delta}\left(\zeta_{0,0}^{(i_1)}+\zeta_{0,1}^{(i_1)}\right)\ \text{a.s.},$$

$$I_{1_{T,t}}^{(i_1)}=-\Delta^{3/2}\left(\frac{1}{2}\zeta_{0,0}^{(i_1)}+\frac{3}{2}\zeta_{0,1}^{(i_1)}+\frac{1}{2\sqrt{3}}\left(\zeta_{1,0}^{(i_1)}+\zeta_{1,1}^{(i_1)}\right)\right)\ \text{a.s.},$$

$$I_{00_{T,t}}^{(i_1 i_2)2,q} = \Delta(\zeta_{0,1}^{(i_2)}\zeta_{0,0}^{(i_1)} + I_{00_{\tau_1,\tau_0}}^{(i_1 i_2)q} + I_{00_{\tau_2,\tau_1}}^{(i_1 i_2)q}),$$

$$I_{000_{T,t}}^{(i_1 i_2 i_3)2,q_1,q_2} = \sqrt{\Delta}(\zeta_{0,1}^{(i_3)} I_{00_{\tau_1,\tau_0}}^{(i_1 i_2)q_1} + \zeta_{0,0}^{(i_1)} I_{00_{\tau_2,\tau_1}}^{(i_2 i_3)q_1}) + I_{000_{\tau_1,\tau_0}}^{(i_1 i_2 i_3)q_2} + I_{000_{\tau_2,\tau_1}}^{(i_1 i_2 i_3)q_2},$$

where $\Delta = (T-t)/2;\ \tau_k = t + k\Delta;\ k = 0, 1, 2$.

Relationships (4), (8)–(12) represent formulae for numerical modeling of stochastic integrals $I_{0_{T,t}}^{(i_1)}$, $I_{1_{T,t}}^{(i_1)}$, $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ by the combined method. Note, that (4), (8)–(12) for $N=1$ become formulae for numerical modeling of the mentioned stochastic integrals by the method of multiple Fourier series. Thus we can state, that the method of multiple Fourier series represents a particular case of the combined method at $N=1$.

Mean-square errors of the combined method of approximation of multiple stochastic integrals

In this section we calculate mean-square errors of approximations, defined by formulae (9), (10).

We have [6]

$$\varepsilon_{N,q} \stackrel{\text{def}}{=} \mathrm{M}\{(I_{00_{T,t}}^{(i_1 i_2)} - I_{00_{T,t}}^{(i_1 i_2)N,q})^2\} = \sum_{k=0}^{N-1} \mathrm{M}\{(I_{00_{\tau_{k+1},\tau_k}}^{(i_1 i_2)} - I_{00_{\tau_{k+1},\tau_k}}^{(i_1 i_2)q})^2\} =$$

$$= N\frac{\Delta^2}{2}\left(\frac{1}{2} - \sum_{l=1}^{q}\frac{1}{4l^2-1}\right) = \frac{(T-t)^2}{2N}\left(\frac{1}{2} - \sum_{l=1}^{q}\frac{1}{4l^2-1}\right), \tag{15}$$

$$\varepsilon_{N,q_1,q_2} \stackrel{\text{def}}{=} \mathrm{M}\{(I_{000_{T,t}}^{(i_1 i_2 i_3)} - I_{000_{T,t}}^{(i_1 i_2)N,q_1,q_2})^2\} =$$

$$= \mathrm{M}\left\{\left(\sum_{k=0}^{N-1}\left(\sqrt{\Delta}\sum_{l=0}^{k-1}(\zeta_{0,k}^{(i_3)}(I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} - I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)q_1}) + \right.\right.\right.$$

$$\left.\left.\left. + \zeta_{0,l}^{(i_1)}(I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)} - I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)q_1})) + I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)} - I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)q_2}\right)\right)^2\right\} =$$

$$= \sum_{k=0}^{N-1}\mathrm{M}\left\{\left(\sqrt{\Delta}\sum_{l=0}^{k-1}(\zeta_{0,k}^{(i_3)}(I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} - I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)q_1}) + \right.\right.$$

$$\left.\left. + \zeta_{0,l}^{(i_1)}(I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)} - I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)q_1})) + I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)} - I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)q_2}\right)^2\right\} =$$

$$= \sum_{k=0}^{N-1}\left(\Delta\mathrm{M}\left\{\left(\zeta_{0,k}^{(i_3)}\sum_{l=0}^{k-1}(I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} - I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)q_1})\right)^2\right\} + \right.$$

$$\left. + \Delta\mathrm{M}\left\{\left((I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)} - I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)q_1})\sum_{l=0}^{k-1}\zeta_{0,l}^{(i_1)}\right)^2\right\} + \delta_{k,q_2}^{(i_1 i_2 i_3)}\right) =$$

$$= \sum_{k=0}^{N-1}\left(\Delta\sum_{l=0}^{k-1}\mathrm{M}\{(I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} - I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)q_1})^2\} + k\Delta\mathrm{M}\{(I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)} - I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)q_1})^2\} + \delta_{k,q_2}^{(i_1 i_2 i_3)}\right) =$$

$$= \sum_{k=0}^{N-1}\left(2k\Delta\sum_{l=0}^{k-1}\mathrm{M}\{(I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} - I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)q_1})^2\} + \delta_{k,q_2}^{(i_1 i_2 i_3)}\right) =$$

$$= \sum_{k=0}^{N-1}\left(2k\Delta\frac{\Delta^2}{2}\left(\frac{1}{2} - \sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right) + \delta_{k,q_2}^{(i_1 i_2 i_3)}\right) =$$

$$= \Delta^3\frac{N(N-1)}{2}\left(\frac{1}{2} - \sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right) + \sum_{k=0}^{N-1}\delta_{k,q_2}^{(i_1 i_2 i_3)} =$$

$$= \frac{1}{2}(T-t)^3\left(\frac{1}{N} - \frac{1}{N^2}\right)\left(\frac{1}{2} - \sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right) + \sum_{k=0}^{N-1}\delta_{k,q_2}^{(i_1 i_2 i_3)}. \tag{16}$$

Here

$$\delta_{k,q_2}^{(i_1 i_2 i_3)} = \mathrm{M}\{(I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)} - I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)q_2})^2\};$$

$i_1 \neq i_2$ in (15) and not all indices i_1, i_2, i_3 in (16) are equal to each other (otherwise there exist exact relationships [2] for modeling integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$).

Let i_1, i_2, i_3 in (16) be pairwise different. Then [6]

$$\delta_{k,q_2}^{(i_1 i_2 i_3)} = \Delta^3\left(\frac{1}{6} - \sum_{i,j,l=0}^{q_2}\frac{C_{ijl}^2}{\Delta^3}\right). \tag{17}$$

Substituting (13) into (17) and (17) into (16), we finally obtain

$$\varepsilon_{N,q_1,q_2} = \frac{1}{2}(T-t)^3\left(\frac{1}{N} - \frac{1}{N^2}\right)\left(\frac{1}{2} - \sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right) +$$

$$+ \frac{(T-t)^3}{N^2}\left(\frac{1}{6} - \sum_{i,j,l=0}^{q_2}\frac{(2i+1)(2j+1)(2l+1)}{64}\bar{C}_{ijl}^2\right). \tag{18}$$

Notice that at $N = 1$ formulae (15), (18) are transformed into corresponding formulae for mean-square errors of approximations of integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$, obtained by the method of multiple Fourier series in Legendre polynomials [6].

Numerical simulations

It is known [2–4], that for numerical implementation of a strong numeric method of accuracy order 1.0 for equation (1) one should numerically model integrals $I_{0_{T,t}}^{(i_1)}$, $I_{00_{T,t}}^{(i_1 i_2)}$ at each step of integration. With this purpose one can use relationships (4), (9), (11). At that, error of approximation of integral $I_{00_{T,t}}^{(i_1 i_2)}$ is determined by formula (15). Let us calculate quantity $\varepsilon_{N,q}$ at different values of N and q:

$$\varepsilon_{3,2} \approx 0.0167(T-t)^2,\ \ \varepsilon_{2,3} \approx 0.0179\,(T-t)^2, \tag{19}$$

$$\varepsilon_{1,6} \approx 0.0192\,(T-t)^2. \tag{20}$$

In all three cases computational efforts for modeling integrals $I_{0_{T,t}}^{(i_1)}$, $I_{00_{T,t}}^{(i_1 i_2)}$ are about the same, however the combined method (formulae (19)) requires calculation of significantly smaller number of Fourier coefficients, than the method of multiple Fourier series (formula (20)).

Let mean-square error of approximation of integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ be equal to $(T-t)^4$. In the Table we list values of N, q, q_1, q_2, that satisfy the system of inequalities

$$\begin{cases} \varepsilon_{N,q} \le (T-t)^4, \\ \varepsilon_{N,q_1,q_2} \le (T-t)^4, \end{cases} \tag{21}$$

and the total number M of Fourier coefficients necessary for approximation of integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ at $T-t = 0.1,\ 0.05,\ 0.02$ (the numbers q, q_1, q_2 were chosen so that the number M would be the smallest).

Table

N	$T-t$											
	0.1				0.05				0.02			
	q	q_1	q_2	M	q	q_1	q_2	M	q	q_1	q_2	M
1	13	—	1	21	50	—	2	77	312	—	6	655
2	6	0	0	7	25	2	0	26	156	4	2	183
3	4	0	0	5	17	1	0	18	104	6	0	105

One can see from the Table, that for small N ($N = 2$) the combined method allows one to reduce significantly the total number of Fourier coefficients necessary for approximation of integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ compared to the method of multiple Fourier series ($N = 1$). At that, computational efforts of the both methods for approximation of integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ are about the same (accuracy of approximation of stochastic integrals for the combined method and the method of multiple Fourier series was taken the same and equal to $(T-t)^4$).

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