Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals

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Abstract  It is well known, that Ito stochastic differential equations (SDE) are adequate mathematical models of dynamic systems under the influence of random disturbances. One of the effective approaches to numerical integration of Ito SDE is an approach based on Taylor-Ito and Taylor-Stratonovich expansions. The most important feature of such expansions is presence in them of so called multiple Ito or Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDE. We successfully use the tool of multiple Fourier series, built in the space $L_2$, to solve the problem of numerical integration of Ito SDE. We successfully use the tool of multiple Fourier series, built in the space $L_2$, to solve the problem of numerical integration of Ito SDE.

Introduction

Let $(\Omega, F, P)$ be a fixed probability space and $W_t$ — is $F_t$-measurable $\forall t \in [0, T]$ Wiener process with independent components $W_t^i$; $i = 1, \ldots, m$. Let’s analyze the following Ito SDE:

$$dX_t = a(X_t, t)dt + B(X_t, t)dW_t, \quad X_0 = X(0, \omega),$$

where $a: \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$, $B: \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m}$ satisfy the standard conditions of existence and uniqueness of strong solution $X_t \in \mathbb{R}^n$ of SDE [99]; $X_0$ and $W_t - W_0 (t > 0)$ are independent. In theorems 1–3 we solve the problem of combined mean-square approximation of stochastic integrals from Taylor-Ito and Taylor-Stratonovich expansions for the process $X_t$.

Main results

**Theorem 1.** Assume, that $\psi_1(t) \in C([t, T]) (i = 1, 2, \ldots, k)$ and $|\phi_j(x)|_{j=0}^\infty$ is a full orthonormal system of continuous functions in the space $L_2([t, T])$. Then

$$J[\psi(t)]_{T,t} = \lim_{n \to \infty} \sum_{i=0}^k \phi_j(t_i) dW_{t_i}$$

where $J[\psi(t)]_{T,t} = \int_t^T \phi_j(t) dW_{t} (t = 0, 1, \ldots, m)$, $W_{t=0} = 0$, $\psi_1(t) = \int_t^T \phi_j(t) dW_{t} (t = 0, 1, \ldots, m)$, $W_{t=0} = 0$, $\phi_j(t) = \int_t^T \phi_j(t) dW_{t} (t = 0, 1, \ldots, m)$, $W_{t=0} = 0$.

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Let’s consider particular cases of the theorem 1 for $k = 1, \ldots$:

\[
\frac{d}{dt} \psi(t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} C_{ij} \dot{\xi}^{(i)}_{j} \dot{\xi}^{(i)}_{j} - \frac{d}{dt} \phi(t) + \frac{d}{dt} \gamma(t),
\]

where $\frac{d}{dt} \psi(t)$ is a truncated series from the theorem 1 with upper limits $p_1, \ldots, p_k$.

The following theorem is adapt to theorem 1 to the multiple Stratonovich stochastic integrals.

**Theorem 3.** Let function $\psi_2(s)$ is continuously differentiated at $t, T$ and functions $\psi_1(s), \psi_3(s)$ are two times continuously differentiated at $t, T$; $\{ \phi_j(x) \}_{j=0}^{\infty}$ is a full orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then

\[
J^* \psi_2(t) = \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_1 j_2} \dot{\xi}^{(j_1)}_{j_1} \dot{\xi}^{(j_2)}_{j_2},
\]

where $J^* \psi_2(t)$ is a multiple Stratonovich stochastic integral; $k = 3, 4$ (for $k = 3$: $i_1, i_2, i_3 = 1, \ldots, m$; for $k = 4$: $i_1, \ldots, i_4 = 0, 1, \ldots, m$ and $\psi_1(t), \psi_2(t) \equiv 1$); the meaning of notations from theorem 1 is remained.

**References**