Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals

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Abstract It is well known, that Ito stochastic differential equations (SDE) are adequate mathematical models of dynamic systems under the influence of random disturbances. One of the effective approaches to numerical integration of Ito SDE is an approach based on Taylor-Ito and Taylor-Stratonovich expansions. The most important feature of such expansions is presence in them of so called multiple Ito or Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDE. We successfully use the tool of multiple Fourier series, built in the space L_2 and poitwise, for the mean-square approximation of multiple stochastic integrals.

Introduction

Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a fixed probability space and $\mathbf{W}_t$ — is F_t-measurable $\forall\, t \in [0, T]$ Wiener process with independent components $\mathbf{W}_t^{(i)};\ i = 1, \ldots, m$. Let's analyze the following Ito SDE:

$$d\mathbf{X}_t = \mathbf{a}(\mathbf{X}_t, t)\,dt + B(\mathbf{X}_t, t)\,d\mathbf{W}_t,\ \mathbf{X}_0 = \mathbf{X}(0, \omega), \tag{99}$$

where $\mathbf{a} : \Re^n \times [0, T] \to \Re^n$, $B : \Re^n \times [0, T] \to \Re^{n\times m}$ satisfy the standard conditions of existence and uniqueness of strong solution $\mathbf{X}_t \in \Re^n$ of SDE (99); $\mathbf{X}_0$ and $\mathbf{W}_t - \mathbf{W}_0 (t > 0)$ — are independent. In theorems 1 – 3 we solve the problem of combined mean-square approximation of stochastic integrals from Taylor-Ito and Taylor-Stratonovich expansions for the prosess $\mathbf{X}_t$.

Main results

Theorem 1. *Assume, that* $\psi_i(\tau) \in C_{[t,T]}$ $(i = 1, 2, \ldots, k)$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *— is a full orthonormal system of continuous functions in the space* $L_2([t, T])$*. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \Bigl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \prod_{s=1}^{k} \phi_{j_{l_s}}(\tau_{l_s}) \Delta\mathbf{W}_{\tau_{l_s}}^{(i_s)}\Bigr),$$

where $J[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \ldots \int_t^{t_2} \psi_1(t_1)\, d\mathbf{W}_{t_1}^{(i_1)} \ldots d\mathbf{W}_{t_k}^{(i_k)}$ *(multiple Ito stochastic integral);* $\Delta\mathbf{W}_{\tau_j}^{(i)} = \mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}$ $(i = 0, 1, \ldots, m)$, $\mathbf{W}_\tau^{(0)} = \tau$, $\zeta_j^{(i)} = \int_t^T \phi_j(\tau)\, d\mathbf{W}_\tau^{(i)}$ *— are independent standard Gaussian random variables for various* i *or* j *if* $i \neq 0$, $\{\tau_j\}_{j=0}^{N-1}$ *— partition of interval* $[t, T]$*, satisfying the conditions:* $t = \tau_0 < \ldots < \tau_N = T$, $\max_{0\le j\le N-1}(\tau_{j+1} - \tau_j) \to 0$ *if* $N \to \infty$, $C_{j_k\ldots j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l)\, dt_1 \ldots dt_k$, $K(t_1, \ldots, t_k) = \mathbf{1}_{\{t_1<\ldots<t_k\}} \psi_1(t_1) \ldots \psi_k(t_k)$ $(t_1, \ldots, t_k \in$

Keywords : *multiple Ito stochastic integral; multiple Stratonovich stochastic inegral; Taylor-Ito expansion; strong approximation; numerical modeling.*

2010 Mathematics Subject Classification : **60H10; 60H05; 60H35; 65C30.**

$[t,T])$, $\mathbf{1}_A$ *— is an indicator of the set A,* $\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k$, $\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\neq l_r(g\neq r);\ g,r=1,\ldots,k\}$, $\mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\}$.
Let's consider particular cases of the theorem 1 for $k=2,3,4:$

$$J[\psi^{(2)}]_{T,t}=\sum_{j_1,j_2=0}^{\infty}C_{j_2j_1}(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\neq 0,j_1=j_2\}}),$$
$$J[\psi^{(3)}]_{T,t}=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\neq 0,j_1=j_2\}}\zeta_{j_3}^{(i_3)}--\mathbf{1}_{\{i_2=i_3\neq 0,j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\neq 0,j_1=j_3\}}\zeta_{j_2}^{(i_2)}),$$
$$J[\psi^{(4)}]_{T,t}=\sum_{j_1,\ldots,j_4=0}^{\infty}C_{j_4\ldots j_1}(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_1=i_2\neq 0,j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\neq 0,j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\neq 0,j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\neq 0,j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_2=i_4\neq 0,j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_3=i_4\neq 0,j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_2\neq 0,j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0,j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\neq 0,j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0,j_2=j_4\}}++\mathbf{1}_{\{i_1=i_4\neq 0,j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0,j_2=j_3\}}).$$

Let's consider the estimates of mean-square errors of approximations, based on theorem 1.
Theorem 2. *In the conditions of the theorem 1 the following estimates are valid:*

$$\mathsf{M}\{(J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}-J[\psi^{(k)}]_{T,t})^2\}\leq k!(\int_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{p_1,\ldots,p_k}C_{j_k\ldots j_1}^2)$$
$$(i_1,\ldots,i_k=0,1,\ldots,m\ \textit{and}\ T-t<1\ \textit{or}\ i_1,\ldots,i_k=1,\ldots,m)$$
$$\textit{and}$$
$$\mathsf{M}\{(J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}-J[\psi^{(k)}]_{T,t})^2\}=\int_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{p_1,\ldots,p_k}C_{j_k\ldots j_1}^2$$
$$(i_1,\ldots,i_k=1,\ldots,m\ \textit{and pairwise different}),$$

where $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is a truncated series from the theorem 1 with upper limits* $p_1,\ldots,p_k$.
The following theorem is adapt theorem 1 to the multiple Stratonovich stochastic integrals.
Theorem 3. *Let function* $\psi_2(s)$ *— is continuously differentiated at* $[t,T]$ *and functions* $\psi_1(s)$, $\psi_3(s)$ *— are two times continuously differentiated at* $[t,T]$; $\{\phi_j(x)\}_{j=0}^{\infty}$ *— is a full orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then*

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},\ J^*[\psi^{(k)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)},$$

where $J^*[\psi^{(k)}]_{T,t}=\int_t^{*T}\psi_k(t_k)\ldots\int_t^{*t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}$ *(multiple Stratonovich stochastic integral);* $k=3,4$ *(for* $k=3$*:* $i_1,i_2,i_3=1,\ldots,m$*; for* $k=4$*:* $i_1,\ldots,i_4=0,1,\ldots,m$ *and* $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1$*); the meaning of notations from theorem* 1 *is remained.*

References

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