Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals

Dmitriy F. Kuznetsov

Peter the Great Saint-Petersburg Polytechnic University, Russia

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http://www.sde-kuznetsov.spb.ru/kuz-2019.pdf

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Contents

1. Examples, Motivation, Problems, Objects of Study

Example 1 (J.M.C. Clark and R.J. Cameron (1980)):

$$d\mathbf{x}_t = A\mathbf{x}_t d\mathbf{w}_t^{(2)} + B d\mathbf{w}_t, \ t \in [t_0, T] \quad \text{(system of Ito SDEs)} \qquad (1)$$

$$\mathbf{x}_t = \begin{pmatrix} \mathbf{x}_t^{(1)} \\ \mathbf{x}_t^{(2)} \end{pmatrix}, \ \mathbf{w}_t = \begin{pmatrix} \mathbf{w}_t^{(1)} \\ \mathbf{w}_t^{(2)} \end{pmatrix}, \ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{x}_{t_0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $\mathbf{w}_t^{(1)}$, $\mathbf{w}_t^{(2)}$ are independent standard Wiener processes.

The exact solution of (1) is:

$$\mathbf{x}_t^{(1)} = \int_{t_0}^{t} d\mathbf{w}_\tau^{(1)}, \ \mathbf{x}_t^{(2)} = \int_{t_0}^{t} \int_{t_0}^{\tau} d\mathbf{w}_s^{(1)} d\mathbf{w}_\tau^{(2)}.$$

The component $\mathbf{x}_t^{(2)}$ can not be expressed in finite form using $N_{i.i.d.}(0,1)$ random variables.

Example 2. System of Ito SDEs:

$$d\mathbf{x_t} = (A\mathbf{x}_t + \mathbf{b}(t))\, dw_t,\ \ t \in [t_0, T], \tag{2}$$

$$\mathbf{x}_t = \begin{pmatrix} \mathbf{x}_t^{(1)} \\ \mathbf{x}_t^{(2)} \\ \mathbf{x}_t^{(3)} \end{pmatrix},\ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \mathbf{b}(t) = \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix},\ \mathbf{x}_{t_0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and w_t is a standard Wiener process.

The exact solution of (2) is:

$$\mathbf{x}_t^{(1)} = \int\limits_{t_0}^{t} dw_\tau,\ \mathbf{x}_t^{(2)} = \int\limits_{t_0}^{t} s dw_s,\ \mathbf{x}_t^{(3)} = \int\limits_{t_0}^{t}\int\limits_{t_0}^{s} \tau dw_\tau dw_s,$$

The component $\mathbf{x}_t^{(3)}$ can not be expressed in finite form using $N_{i.i.d.}(0,1)$ random variables.

Example 3 (P.E. Kloeden and E. Platen (1992)). System of Ito SDEs:

$$d\mathbf{x}_t = (A\mathbf{x}_t + \mathbf{g}(t))\, dt + \sum_{j=1}^{m} B_j \mathbf{x}_t d\mathbf{w}_t^{(j)} + C(t) d\mathbf{w}_t, \ \ t \in [t_0, T], \qquad (3)$$

$$\mathbf{x}_t = \begin{pmatrix} \mathbf{x}_t^{(1)} \\ \ldots \\ \mathbf{x}_t^{(n)} \end{pmatrix}, \ \ \mathbf{w}_t = \begin{pmatrix} \mathbf{w}_t^{(1)} \\ \ldots \\ \mathbf{w}_t^{(m)} \end{pmatrix}, \ \ \mathbf{g} : [t_0, T] \to \Re^n, C : [t_0, T] \to \Re^{n\times m},$$

$$A, B_1, \ldots, B_m \in \Re^{n\times n},$$

$$AB_j = B_j A, \ \ B_i B_j = B_j B_i; \ \ i, j = 1, \ldots, m,$$

and $\mathbf{w}_t^{(1)}, \ldots, \mathbf{w}_t^{(m)}$ are independent standard Wiener processes.

The exact solution of (3) is:

$$\mathbf{x}_t = X_{t,t_0}\Biggl(\mathbf{x}_{t_0} + \int\limits_{t_0}^{t} X_{s,t_0}^{-1}\Biggl(\mathbf{g}(s) - \sum_{j=1}^{m} B_j \mathbf{c}_j(s)\Biggr)ds + \int\limits_{t_0}^{t} X_{s,t_0}^{-1} C(s) d\mathbf{w}_s\Biggr),$$

where

$$X_{t,t_0} = \exp\left\{\Biggl(A - \frac{1}{2}\sum_{j=1}^{m} B_j^2\Biggr)(t - t_0) + \sum_{j=1}^{m} B_j \left(\mathbf{w}_t^{(j)} - \mathbf{w}_{t_0}^{(j)}\right)\right\},$$

and $\mathbf{c}_j(t)$ is j-th colomn of matrix $C(t)$.

The small value $t - t_0$ is the integration step for numerical procedures for Ito SDEs, so we do not want to use additional partition of $[t_0, t]$. Otherwise, it results in the growth of computational errors and a significant increase in computational costs.

If we use the formula $\exp\{G\} \approx I + G$, then the stochastic integral

$$\int\limits_{t_0}^{t} X_{s,t_0}^{-1} C(s) d\mathbf{w}_s$$

leads to iterated Ito stochastic integrals:

$$\int\limits_{t_0}^{t} \psi_1(s) d\mathbf{w}_s^{(i)}, \quad \int\limits_{t_0}^{t} \psi_2(s) \int\limits_{t_0}^{s} d\mathbf{w}_\tau^{(j)} d\mathbf{w}_s^{(i)} \qquad (i, j = 1, \ldots, m),$$

where $\psi_1(s), \psi_2(s) : [t_0, T] \to \Re^1$ are non-random functions.

Thus, the knowledge of the exact solution of Ito SDE does not solve the problem of the numerical solution of Ito SDE or the visualization of sample paths of the solution of Ito SDE.

Let $(\Omega, \mathbf{F}, \mathbf{F}_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space, and $\mathbf{w}_t$, $t \in [0,T]$ be a $\mathbf{F}_t$-measurable for all $t \in [0,T]$ m-dimensional standard Wiener process with independent components $\mathbf{w}_t^{(1)}, \ldots, \mathbf{w}_t^{(m)}$.

Example 4. System of non-linear Ito SDEs:

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{a}(\mathbf{x}_s, s)ds + \sum_{i=1}^{m} \int_0^t \mathbf{b}_i(\mathbf{x}_s, s)d\mathbf{w}_s^{(i)}, \ \mathbf{x}_0 = \mathbf{x}(0, \omega),$$

where smooth functions $\mathbf{a}, \mathbf{b}_1, \ldots, \mathbf{b}_m : \Re^n \times [0,T] \to \Re^n$ satisfy the standard conditions of existence and uniqueness of strong solution $\mathbf{x}_t \in \Re^n$; $\mathbf{x}_0$ and $\mathbf{w}_t - \mathbf{w}_0$ $(t > 0)$ are independent; $\mathbf{E}|\mathbf{x}_0|^2 < \infty$; $\mathbf{E}$ is the expectation operator.

From iterated application of Ito formula for $t > t_0$ we have (E. Platen and W. Wagner (1982)):

$$\mathbf{x}_t = \mathbf{x}_{t_0} + \sum_{i_1=1}^{m} \mathbf{b}_{i_1}(\mathbf{x}_{t_0}, t_0) \int_{t_0}^{t} d\mathbf{w}_{t_1}^{(i_1)} + \mathbf{a}(\mathbf{x}_{t_0}, t_0) \int_{t_0}^{t} dt_1 + \qquad (4)$$

$$+ \sum_{i_1,i_2=1}^{m} G_{i_2}\mathbf{b}_{i_1}(\mathbf{x}_{t_0}, t_0) \int_{t_0}^{t} \int_{t_0}^{t_1} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_1}^{(i_1)} + \qquad (5)$$

$$+ \sum_{i_1=1}^{m} \left(G_{i_1}\mathbf{a}(\mathbf{x}_{t_0}, t_0) \int_{t_0}^{t} \int_{t_0}^{t_1} d\mathbf{w}_{t_2}^{(i_1)} dt_1 + L\mathbf{b}_{i_1}(\mathbf{x}_{t_0}, t_0) \int_{t_0}^{t} \int_{t_0}^{t_1} dt_2 d\mathbf{w}_{t_1}^{(i_1)} \right) +$$

$$+ \sum_{i_1,i_2,i_3=1}^{m} G_{i_3}G_{i_2}\mathbf{b}_{i_1}(\mathbf{x}_{t_0}, t_0) \int_{t_0}^{t} \int_{t_0}^{t_1} \int_{t_0}^{t_2} d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_1}^{(i_1)} + \ldots \text{ w. p. 1},$$

where L, G_i $(i = 1, \ldots, m)$ are differential operators from the Ito formula, (4) — Euler method, (4)+(5) — Milstein method.

Example 5. Numerical simulation of mild solutions of semilinear stochastic partial differential equations (SPDEs).

Define infinite dimensional Gaussian process (so-called Q-Wiener process) as follows

$$\mathbf{W}(t,x)=\sum_{i\in J,\lambda_i\neq 0} e_i(x)\sqrt{\lambda_i}\mathbf{w}_t^{(i)}, \quad \mathbf{W}^M(t,x)=\sum_{i\in J_M,\lambda_i\neq 0} e_i(x)\sqrt{\lambda_i}\mathbf{w}_t^{(i)},$$

where $t\in[0,T]$, $x\in \mathrm{D}$, $i=(i_1,\ldots,i_d)$, $x=(x_1,\ldots,x_d)$, $J=\mathbf{N}^d$, $\lambda_i\geq 0$, $e_i(x)$ is an orthonormal basis in $L_2(\mathrm{D})$, $J_M=\{i:|i|\leq M\}$, $\langle e_i,\mathbf{W}_t\rangle/\sqrt{\lambda_i}=\mathbf{w}_t^{(i)}$ for $\lambda_i\neq 0$, $\mathbf{w}_t^{(i)}$ are independent standard Wiener processes, $\langle\cdot,\cdot\rangle$ is a scalar product in $L_2(\mathrm{D})$, $\mathbf{W}_t$ is $\mathbf{W}(t,x)$.

S. Becker, A. Jentzen and P.E. Kloeden, "An exponential Wagner-Platen type scheme for SPDEs" (2016). This scheme contains the following set of iterated Ito stochastic integrals:

$$\int_{t_0}^{t}\ldots\int_{t_0}^{t_2} d\langle e_{i_1},\mathbf{W}_{t_1}\rangle\ldots d\langle e_{i_k},\mathbf{W}_{t_k}\rangle, \quad i_1,\ldots,i_k\in J_M, \quad k=1,2,3.$$

Consider the partition of interval $[0, T]$ for which

$$0 = \tau_0 < \ldots < \tau_N = T, \;\; \Delta_N = \max_{0 \leq j \leq N-1} |\tau_{j+1} - \tau_j|.$$

Definition. *We shall say that time discrete approximation* $\mathbf{y}_j$ $(j = 0, 1, \ldots, N)$ *converges in the mean-square sense with order* $\gamma > 0$ *if there exists a constant* $C > 0$, *which does not depend on* Δ_N, *and* $\delta > 0$ *such that*

$$\left(\mathbf{E}|\mathbf{x}_{\tau_j} - \mathbf{y}_j|^2\right)^{1/2} \leq C(\Delta_N)^{\gamma} \quad \forall \Delta_N \in (0, \delta).$$

P.E. Kloeden and A. Neuenkirch (2007):

$$\left(\mathbf{E} \sup_{0 \leq j \leq N} |\mathbf{x}_{\tau_j} - \mathbf{y}_j|^p\right)^{1/p} \leq C(\Delta_N)^{\gamma} \, \forall p \geq 1 \Rightarrow \sup_{0 \leq j \leq N} |\mathbf{x}_{\tau_j} - \mathbf{y}_j| \leq \eta_{\varepsilon,\gamma}(\Delta_N)^{\gamma - \varepsilon}$$

$\forall \varepsilon > 0$ w. p. 1, where $\eta_{\varepsilon,\gamma}$ is a random variable with all moments finite.

So, our goal is mean-square approximation and approximation of p th mean of iterated Ito and Stratonovich stochastic integrals in the form:

$$I[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \ldots \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},$$

$$S[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \ldots \int_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i_1)} \ldots \circ d\mathbf{w}_{t_k}^{(i_k)},$$

where $I[\psi^{(k)}]_{T,t}$ — Ito stochastic integral, $S[\psi^{(k)}]_{T,t}$ — Stratonovich stochastic integral; $i_1, \ldots, i_k = 0, 1, \ldots, m$; $\mathbf{w}_\tau^{(1)}, \ldots, \mathbf{w}_\tau^{(m)}$ are independent standard Wiener processes; $\mathbf{w}_\tau^{(0)} = \tau$; functions $\psi_1(\tau), \ldots, \psi_k(\tau)$ are continuous or smooth non-random functions at the interval $[t, T]$.

2. Expansion of Iterated Ito Stochastic Integrals

Denotations

Assume that $\psi_1(\tau), \ldots, \psi_k(\tau)$ are continuous non-random functions at the interval $[t, T]$,

$$K(t_1, \ldots, t_k) = \psi_1(t_1) \ldots \psi_k(t_k) \mathbf{1}_{\{t_1 < \ldots < t_k\}} \ (k \geq 2),$$

$K(t_1) = \psi_1(t_1)$, $\mathbf{1}_A$ is an indicator of the set A, $\phi_j(x)$ is a complete orthonormal system of functions in the space $L_2([t, T])$,

$$C_{j_k \ldots j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \phi_{j_1}(t_1) \ldots \phi_{j_k}(t_k) dt_1 \ldots dt_k$$

is a Fourier coefficient, $N_{i.i.d.}(0, 1)$ are independent standard Gaussian random variables, l.i.m. is a limit in the mean-square sense,

$$t = \tau_0 < \ldots < \tau_N = T, \ \Delta_N = \max_{0 \leq l \leq N-1} |\tau_{l+1} - \tau_l| \to 0 \text{ if } N \to \infty,$$

$$\Delta \mathbf{w}_{\tau_l}^{(i)} = \mathbf{w}_{\tau_{l+1}}^{(i)} - \mathbf{w}_{\tau_l}^{(i)} \ (l = 0, 1, \ldots, N-1).$$

Theorem 1 (2006, [1]). *Assume that* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous non-random functions at the interval* $[t,T]$, $\phi_j(x)$ *is a complete orthonormal system of continuous functions in the space* $L_2([t,T])$, *and* $i_1,\ldots,i_k=0,1,\ldots,m$. *Then*

$$I[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k},$$

$$I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathcal{H}_k\backslash\mathcal{L}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr),\qquad(6)$$

$$\mathcal{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,1,\ldots,N-1\},$$

$$\mathcal{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=\overline{1,k}\},$$

$$\zeta_j^{(i)}=\int_t^T\phi_j(s)d\mathbf{w}_s^{(i)}\sim N_{i.i.d.}(0,1)\ \textit{for various } i \textit{ or } j\ \textit{(if } i\ne 0).$$

Sketch of proof for the case $k=2$; $i_1, i_2 = 1, \ldots, m$ (2006, [1]):

$$\int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,l_2=0\\ l_1\ne l_2}}^{N-1} K(\tau_{l_1}, \tau_{l_2}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} \stackrel{(*)}{=}$$

$$K(t_1,t_2) = S + \underbrace{K(t_1,t_2) - S}_{R_{p_1 p_2}(t_1,t_2)}, \quad S = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_2)$$

$$\stackrel{(*)}{=} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{l_1,l_2=0}^{N-1} \phi_{j_1}(\tau_{l_1}) \phi_{j_2}(\tau_{l_2}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)} - \right.$$

$$\left. - \sum_{l_1=0}^{N-1} \phi_{j_1}(\tau_{l_1}) \phi_{j_2}(\tau_{l_1}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)} \right) +$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{l_1,l_2=0\\ l_1\ne l_2}}^{N-1} R_{p_1,p_2}(\tau_{l_1}, \tau_{l_2}) \Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}$$

$$\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_1,l_2=0}^{N-1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}=\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

$$\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{l_1=0}^{N-1}\phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}=\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\phi_{j_1}(\tau)\phi_{j_2}(\tau)d\tau=$$

$$=\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}},$$

$$\underset{N\to\infty}{\text{l.i.m.}}\ \sum_{\substack{l_1,l_2=0\\ l_1\neq l_2}}^{N-1}R_{p_1,p_2}(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}=\underset{N\to\infty}{\text{l.i.m.}}\ \left(\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{l_2-1}+\sum_{l_1=0}^{N-1}\sum_{l_2=0}^{l_1-1}\right)=$$

$$=\int_t^T\int_t^{t_2}R_{p_1,p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+$$

$$+\int_t^T\int_t^{t_1}R_{p_1,p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}$$

$$\mathbf{E}\Bigl(\int_t^T\int_t^{t_2} R_{p_1,p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int_t^T\int_t^{t_1} R_{p_1,p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}\Bigr)^2\le$$

$$\le 2\Bigl(\int_t^T\int_t^{t_2} R^2_{p_1,p_2}(t_1,t_2)dt_1dt_2+\int_t^T\int_t^{t_1} R^2_{p_1,p_2}(t_1,t_2)dt_2dt_1\Bigr)=$$

$$=2\Bigl(\int_t^T\int_t^{t_2} R^2_{p_1,p_2}(t_1,t_2)dt_1dt_2+\int_t^T\int_{t_2}^{T} R^2_{p_1,p_2}(t_1,t_2)dt_2dt_1\Bigr)=$$

$$=2\int_{[t,T]^2} R^2_{p_1,p_2}(t_1,t_2)dt_1dt_2=$$

$$=2!\Bigl(\|K\|^2_{L_2([t,T]^2)}-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C^2_{j_2j_1}\Bigr)\to 0 \ \text{as} \ p_1,p_2\to\infty$$

Particular cases of the theorem 1 for $k=1,\ldots,5$ (2006, [1]):

$$
I[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)},
$$

$$
I[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right),
$$

$$
I[\psi^{(3)}]_{T,t}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-
$$

$$
-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),
$$

$$
I[\psi^{(4)}]_{T,t} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),
$$

$$I[\psi^{(5)}]_{T,t} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\neq 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\neq 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\neq 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr).$$

For $\psi_1(\tau),\ldots,\psi_5(\tau) \equiv \psi(\tau)$, and $i_1 = \ldots = i_5 = 1,\ldots,m$ w.p.1 from the theorem 1 we get ($k = 2, 3$ (2007, 2009, [1])):

$$I[\psi^{(1)}]_{T,t} = \frac{1}{1!}\delta_{T,t},$$

$$I[\psi^{(2)}]_{T,t} = \frac{1}{2!}\left(\delta_{T,t}^2 - \Delta_{T,t}\right),$$

$$I[\psi^{(3)}]_{T,t} = \frac{1}{3!}\left(\delta_{T,t}^3 - 3\delta_{T,t}\Delta_{T,t}\right),$$

$$I[\psi^{(4)}]_{T,t} = \frac{1}{4!}\left(\delta_{T,t}^4 - 6\delta_{T,t}^2\Delta_{T,t} + 3\Delta_{T,t}^2\right),$$

$$I[\psi^{(5)}]_{T,t} = \frac{1}{5!}\left(\delta_{T,t}^5 - 10\delta_{T,t}^3\Delta_{T,t} + 15\delta_{T,t}\Delta_{T,t}^2\right),$$

where

$$\delta_{T,t} = \int_t^T \psi(s)d\mathbf{w}_s^{(i)},\ \ \Delta_{T,t} = \int_t^T \psi^2(s)ds.$$

Theorem 2. *Assume, that conditions of the theorem 1 be satysfied. Then* $\forall\ T-t \in (0,+\infty)$; $i_1,\ldots,i_k = 1,\ldots,m$ *or* $\forall\ T-t \in (0,1)$; $i_1,\ldots,i_k = 0,1,\ldots,m$ (2017, [1]):

$$\mathbf{E}\left(I[\psi^{(k)}]_{T,t} - I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}\right)^2 \le k!\left(\|K\|^2_{L_2([t,T]^k)} - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}\right).$$

Moreover, $\forall\ T-t \in (0,+\infty)$ *and pairwise different* $i_1,\ldots,i_k = 1,\ldots,m$ (1996, [1]):

$$\mathbf{E}\left(I[\psi^{(k)}]_{T,t} - I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}\right)^2 = \|K\|^2_{L_2([t,T]^k)} - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1},$$

where $I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}$ *is defined by* (6) (*see theorem* 1), *and*

$$\|K\|_{L_2([t,T]^k)} = \left(\int_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Theorem 3. *Assume, that conditions of the theorem 1 be satysfied for* $i_1,\ldots,i_k=1,\ldots,m$. *Then* $\forall\ T-t\in(0,+\infty)$, $n\in\mathbf{N}$ (2007, [1]):

$$\mathbf{E}\Bigl(I[\psi^{(k)}]_{T,t}-I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}\Bigr)^{2n}\ \to 0\ \ \textit{as}\ \ p_1,\ldots,p_k\to\infty,$$

and

$$\mathbf{E}\left(I[\psi^{(k)}]_{T,t}-I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}\right)^{2n}\le C_{n,k}\left(\|K\|^2_{L_2([t,T]^k)}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\right)^n,$$

where

$$C_{n.k}=(k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!$$

Moreover, $\forall\ T-t\in(0,1)$ (2018, [1]):

$$I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}\to I[\psi^{(k)}]_{T,t}\ \textit{as}\ p_1,\ldots,p_k\to\infty\ \textit{w. p. 1.}$$

Theorem 4 (2017, [1-4]). *Assume, that conditions of the theorem 1 be satysfied for* $i_1,\ldots,i_k=1,\ldots,m.$ *Then*

$$
{\bf E}\biggl(I[\psi^{(k)}]_{T,t}-I[\psi^{(k)}]_{T,t}^{p}\biggr)^2=\|K\|^2_{L_2([t,T]^k)}-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\times
$$

$$
\times{\bf E}\left(I[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int_t^T\phi_{j_k}(t_k)\ldots\int_t^{t_2}\phi_{j_1}(t_1)d{\bf w}_{t_1}^{(i_1)}\ldots d{\bf w}_{t_k}^{(i_k)}\right),
$$

where

$$
I[\psi^{(k)}]_{T,t}^{p}=I[\psi^{(k)}]_{T,t}^{p_1\ldots p_k}\biggr|_{p_1,\ldots,p_k=p},
$$

and $\sum\limits_{(j_1,\ldots,j_k)}$ *gives the sum over all possible permutations* $(j_1,\ldots,j_k)$*; moreover, if elements* j_r *and* j_q *in a permutation* $(j_1,\ldots,j_k)$ *swap their places, then the elements* i_r *and* i_q *will do the same in the permutation* $(i_1,\ldots,i_k)$*; the other notations are like in the theorem* 1.

Examples. For $i_1,\ldots,i_4=1,\ldots,m$ from the theorem 4 we have:

$$\mathbf{E}\Big(I[\psi^{(2)}]_{T,t}-I[\psi^{(2)}]_{T,t}^p\Big)^2=$$

$$=\|K\|^2_{L_2([t,T]^2)}-\sum_{j_2,j_1=0}^{p}C_{j_2j_1}\Big(C_{j_2j_1}+C_{j_1j_2}\Big)\ \ (i_1=i_2),$$

$$\mathbf{E}\Big(I[\psi^{(3)}]_{T,t}-I[\psi^{(3)}]_{T,t}^p\Big)^2=$$

$$=\|K\|^2_{L_2([t,T]^3)}-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}\Big(C_{j_3j_2j_1}+C_{j_3j_1j_2}\Big)\ \ (i_1=i_2\ne i_3),$$

$$\mathbf{E}\Big(I[\psi^{(4)}]_{T,t}-I[\psi^{(4)}]_{T,t}^p\Big)^2=\|K\|^2_{L_2([t,T]^4)}-$$

$$-\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\left(\sum_{(j_3,j_4)}\left(\sum_{(j_1,j_2)}C_{j_4j_3j_2j_1}\right)\right)\ \ (i_1=i_2\ne i_3=i_4).$$

Modifications of the theorem 1 (2006, 2018 [1]).

(1) $\phi_j(x)$ is a complete orthonormal system of functions in $L_2([t,T])$, and $\phi_j(\tau)$ $(\forall j < \infty)$ has not more than finite number of points of finite discontinuity and continuous from the right on $[t,T]$ (for example, Haar and Rademacher-Walsh functions).

(2) Wiener processes $\mathbf{w}_s^{(i)}$ $(i = 1, \ldots, m)$ can be replaced by independent martingale Poisson measures.

(3) Wiener processes $\mathbf{w}_s^{(i)}$ $(i = 1, \ldots, m)$ can be replaced by independent martingales $\mathbf{m}_s^{(i)}$ $(i = 1, \ldots, m)$ such that

$$\mathbf{E}\left|\mathbf{m}_s - \mathbf{m}_\tau\right|^2 = \int_\tau^s \rho(x)dx$$

$(\rho(x) \geq 0$ is a continuousy differentiable non-random function on $[t,T])$,

$$\mathbf{E}\left|\mathbf{m}_s - \mathbf{m}_\tau\right|^p \leq C_p|s-\tau|;\ p = 3, 4, \ldots;$$

$\tau, s \in [t,T]$.

(4) The case (3), and $\phi_j(x)$ is a system from the case (1) but with weight $r(x) \geq 0$, and $\rho(x)/r(x) < \infty$.

3. Expansion of Iterated Stratonovich Stochastic Integrals

Let $C_{t,T}^q$ be the space of q times continuousy differentiable functions on $[t,T]$.

Let $P_{t,T}$ be the system of orthonormal Legendre polynomials on $[t,T]$ and $H_{t,T}$ is analogous trigonometric system.

Theorem 5 (2010-2018, [1-4]). *Let conditions of the theorem* 1 *be satysfied. Then*

$$S[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}$$

for each of the following cases:

(1) $k=2$; $i_1,i_2=1,\ldots,m$; $\phi_j(x)$ *is* $P_{t,T}$ *or* $H_{t,T}$; $\psi_2(\tau)\in C_{t,T}^1$; $\psi_1(\tau)\in C_{t,T}^2$.

(2) $k=2$; $i_1,i_2=1,\ldots,m$; $\phi_j(x)$ *is* $P_{t,T}$ *or* $H_{t,T}$; $\psi_1(\tau),\psi_2(\tau)\in C_{t,T}^1$.

(3) $k=3$; $i_1,i_2,i_3=0,1,\ldots,m$; $\phi_j(x)$ *is* $P_{t,T}$ *or* $H_{t,T}$; $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)\equiv 1$.

(4) $k=3$; $i_1,i_2,i_3=1,\ldots,m$; $\phi_j(x)$ is $P_{t,T}$; $\psi_i(\tau)=(t-\tau)^{l_i}$ $(i=1,2,3)$, and one of the following cases takes place:

(a) $i_1\neq i_2,\ i_2\neq i_3,\ i_1\neq i_3$ and $l_1,\ l_2,\ l_3=0,\ 1,\ 2,\ldots$;
(b) $i_1=i_2\neq i_3$ and $l_1=l_2\neq l_3$ and $l_1,\ l_2,\ l_3=0,\ 1,\ 2,\ldots$;
(c) $i_1\neq i_2=i_3$ and $l_1\neq l_2=l_3$ and $l_1,\ l_2,\ l_3=0,\ 1,\ 2,\ldots$;
(d) $i_1,i_2,i_3=1,\ldots,m$; $l_1=l_2=l_3=l$ and $l=0,\ 1,\ 2,\ldots$.

(5) $k=3$; $i_1,i_2,i_3=1,\ldots,m$; $\phi_j(x)$ is $P_{t,T}$ or $H_{t,T}$; $\psi_2(s)\in C^1_{t,T}$; $\psi_1(s),\psi_3(s)\in C^2_{t,T}$; $p_1=p_2=p_3=p$.

(6) $k=3$; $i_1,i_2,i_3=1,\ldots,m$; $\phi_j(x)$ is $P_{t,T}$ or $H_{t,T}$; $\psi_1(\tau),\psi_2(\tau)$, $\psi_3(\tau)\in C^1_{t,T}$; $p_1=p_2=p_3=p$, and one of the following cases takes place:

(a) $i_1\neq i_2,\ i_2\neq i_3,\ i_1\neq i_3$;
(b) $i_1=i_2\neq i_3$, and $\psi_1(s)\equiv\psi_2(s)$;
(c) $i_1\neq i_2=i_3$, and $\psi_2(s)\equiv\psi_3(s)$;
(d) $i_1,i_2,i_3=1,\ldots,m$, and $\psi_1(s)\equiv\psi_2(s)\equiv\psi_3(s)$.

(7)–(8) $k=4,5$; $i_1,\ldots,i_5=0,1,\ldots,m$; $\phi_j(x)$ is $P_{t,T}$ or $H_{t,T}$; $\psi_1(\tau),\ldots,\ \psi_5(\tau)\equiv 1$; $p_1=\ldots=p_5=p$.

Another Approach.

Theorem 6 (1997, [1]). *Let* $\phi_j(x)$ *is* $P_{t,T}$ *of* $H_{t,T}$, *and* $\psi_1(\tau), \ldots, \psi_k(\tau) \in C^1_{t,T}$. *Then* $\forall\ n \in \mathbf{N}$, *and* $i_1, \ldots, i_k = 0, 1, \ldots, m:$

$$S[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)}, \qquad (7)$$

where (7) means that

$$\lim_{p_1 \to \infty} \ldots \lim_{p_k \to \infty} \mathbf{E}\Bigl(S[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)}\Bigr)^{2n} = 0,$$

where

$$S[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \ldots \int_t^{t_2} \psi_1(t_1) \circ d\mathbf{w}_{t_1}^{(i_1)} \ldots \circ d\mathbf{w}_{t_k}^{(i_k)},$$

and

$$\zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{w}_s^{(i)} \sim N_{i.i.d.}(0, 1) \text{ for various } i \text{ or } j \text{ (if } i \neq 0);$$

the other notations are like in the theorem 1.

4. Legendre Polynomials or Trigonometry? Mean-Square Approximation of Concrete Iterated Ito and Stratonovich Stochastic Integrals.

Strong Ito-Taylor schemes of order of accuracy 1.5 for Ito SDEs include 5 iterated Ito stochastic integrals:

$$I_{(1)T,t}^{(i_1)} = \int_t^T d\mathbf{w}_{t_1}^{(i_1)},$$

$$I_{(10)T,t}^{(i_1 0)} = \int_t^T \int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} dt_2, \ \ I_{(01)T,t}^{(0 i_2)} = \int_t^T \int_t^{t_2} dt_1 d\mathbf{w}_{t_2}^{(i_2)},$$

$$I_{(11)T,t}^{(i_1 i_2)} = \int_t^T \int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)}, I_{(111)T,t}^{(i_1 i_2 i_3)} = \int_t^T \int_t^{t_3} \int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)},$$

where $i_1, i_2, i_3 = 1, \ldots, m.$

The case of Legendre polynomials:

$$I_{(1)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)}, \quad I_{(01)T,t}^{(0i_1)} = \frac{(T-t)^{3/2}}{2}\Bigl(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\Bigr), \tag{8}$$

$$I_{(10)T,t}^{(i_1 0)} = \frac{(T-t)^{3/2}}{2}\Bigl(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\Bigr), \tag{9}$$

and (1997, 2006, [1]):

$$I_{(11)T,t}^{(i_1 i_2)q} = \frac{T-t}{2}\Bigl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\Bigl(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\Bigr) - \mathbf{1}_{\{i_1=i_2\}}\Bigr),$$

$$\zeta_j^{(i)} = \int_t^T \phi_j(s)d\mathbf{w}_s^{(i)} \sim N_{i.i.d.}(0,1) \text{ for various } i \text{ or } j \ (i=1,\ldots,m),$$

$I_{(11)T,t}^{(i_1 i_2)} = \underset{q\to\infty}{\text{l.i.m.}}\ I_{(11)T,t}^{(i_1 i_2)q}$, $\phi_j(s)$ is $P_{t,T}$. Moreover (2006, [1]):

$$I_{(111)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1} \Biggl(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \Biggr),$$

$$T - t \ll 1 \Rightarrow q_1 \ll q,$$

$$I_{(111)T,t}^{(i_1 i_2 i_3)} = \underset{q_1 \to \infty}{\text{l.i.m.}}\ I_{(111)T,t}^{(i_1 i_2 i_3)q_1}, \qquad I_{(111)T,t}^{(i_1 i_1 i_1)} = \frac{1}{6}(T-t)^{3/2} \left(\left(\zeta_0^{(i_1)} \right)^3 - 3\zeta_0^{(i_1)} \right),$$

$$C_{j_3 j_2 j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8} \bar{C}_{j_3 j_2 j_1},$$

$$\bar{C}_{j_3 j_2 j_1} = \int_{-1}^{1} P_{j_3}(z) \int_{-1}^{z} P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x) dx dy dz \ \text{(not depends on } t, T\text{)}$$

Denote

$$E_2^q = \mathbf{E}\left(I_{(11)T,t}^{(i_1 i_2)} - I_{(11)T,t}^{(i_1 i_2)q}\right)^2,$$

$$E_3^{q_1} = \mathbf{E}\left(I_{(111)T,t}^{(i_1 i_2 i_3)} - I_{(111)T,t}^{(i_1 i_2 i_3)q_1}\right)^2.$$

Then from the theorem 4 we have (1997, 2007, 2017 [1]):

$$E_2^q = \frac{(T-t)^2}{2}\Bigl(\frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2-1}\Bigr),$$

$$E_3^{q_1} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_2 j_1}^2 \ \ (i_1 \ne i_2, i_1 \ne i_3, i_2 \ne i_3),$$

$$E_3^{q_1} = \frac{(T-t)^3}{6} - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_3 j_2 j_1}^2 - \sum_{j_3,j_2,j_1=0}^{q_1} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1} \ \ (i_1 \ne i_2 = i_3),$$

$$E_3^{q_1}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}C_{j_1j_2j_3}$$

$$(i_1=i_3\ne i_2),$$

$$E_3^{q_1}=\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}^2-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_1j_2}C_{j_3j_2j_1}$$

$$(i_1=i_2\ne i_3).$$

Form the other hand from the theorem 2 (2017, [1]):

$$E_3^{q_1}\le 6\Bigl(\frac{(T-t)^3}{6}-\sum_{j_3,j_2,j_1=0}^{q_1}C_{j_3j_2j_1}^2\Bigr).$$

Coefficients $\bar{C}_{j_3j_2j_1}$ can be calculated exactly via computer algebra packs such as DERIVE, MAPLE, etc.:

Coefficients $\bar{C}_{3jk}$ (1999, [1])

j \ k	0	1	2	3	4	5	6
0	0	$\frac{2}{105}$	0	$-\frac{4}{315}$	0	$\frac{2}{693}$	0
1	$\frac{4}{105}$	0	$-\frac{2}{315}$	0	$-\frac{8}{3465}$	0	$\frac{10}{9009}$
2	$\frac{2}{35}$	$-\frac{2}{105}$	0	$\frac{4}{3465}$	0	$-\frac{74}{45045}$	0
3	$\frac{2}{315}$	0	$-\frac{2}{3465}$	0	$\frac{16}{45045}$	0	$-\frac{10}{9009}$
4	$-\frac{2}{63}$	$\frac{46}{3465}$	0	$-\frac{32}{45045}$	0	$\frac{2}{9009}$	0
5	$-\frac{10}{693}$	0	$\frac{38}{9009}$	0	$-\frac{4}{9009}$	0	$\frac{122}{765765}$
6	0	$-\frac{10}{3003}$	0	$\frac{20}{9009}$	0	$-\frac{226}{765765}$	0

Coefficients $\bar{C}_{100lr}$ (1999, [1])

l \ r	0	1
0	$\frac{8}{45}$	$-\frac{4}{35}$
1	$-\frac{16}{315}$	$\frac{2}{45}$

Minimal numbers q, q_1 for E_2^q, $E_3^{q_1} \le (T-t)^4$ (2006, [1]).

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6

$q_1 \ll q$

$$I_{(11)T,t}^{(i_1 i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right),$$

$$I_{(111)T,t}^{(i_1 i_2 i_3)q_1} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr).$$

The small value $\Delta \ll 1$ is a step of integration for numerical procedures for Ito SDEs.

Examples (1999, 2006, [1]):

$$\mathbf{E}\left(I^{(i_1i_2i_3)}_{(111)t+\Delta,t}-I^{(i_1i_2i_3)6}_{(111)t+\Delta,t}\right)^2=$$

$$=\frac{\Delta^3}{6}-\sum_{i,j,k=0}^{6}C^2_{kji}\approx 0.01956000\Delta^3,$$

$$\mathbf{E}\left(I^{(i_1i_2i_3i_4i_5)}_{(11111)t+\Delta,t}-I^{(i_1i_2i_3i_4i_5)1}_{(11111)t+\Delta,t}\right)^2=$$

$$=\frac{\Delta^5}{120}-\sum_{i,j,k,l,r=0}^{1}C^2_{rlkji}\approx 0.00759105\Delta^5.$$

The case of trigonometric functions (G.N. Milstein (1988) or theorems 1, 6 (1997, 2006, [1])):

$$I_{(10)T,t}^{(i_1 0)q} = \frac{(T-t)^{3/2}}{2}\Biggl(\zeta_0^{(i_1)} + \frac{\sqrt{2}}{\pi}\Bigl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\Bigr)\Biggr), \qquad (10)$$

$$I_{(01)T,t}^{(0 i_1)q} = \frac{(T-t)^{3/2}}{2}\Biggl(\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\Bigl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\Bigr)\Biggr), \qquad (11)$$

$$I_{(1)T,t}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},\ \ \xi_q^{(i)} = \frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\ \zeta_{2r-1}^{(i)},\ \ \alpha_q = \frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2},$$

$\xi_q^{(i)},\ \zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{w}_s^{(i)} \sim N_{i.i.d.}(0,1)$ for various i or j $(i = \overline{1,m})$, $\phi_j(s)$ is $H_{t,T}$. Moreover:

$$I^{(i_2 i_1)q}_{(11)T,t} = \frac{1}{2}(T-t)\Biggl(\zeta^{(i_1)}_0\zeta^{(i_2)}_0 + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\Biggl(\zeta^{(i_1)}_{2r}\zeta^{(i_2)}_{2r-1} - \zeta^{(i_1)}_{2r-1}\zeta^{(i_2)}_{2r} +$$

$$+\sqrt{2}\left(\zeta^{(i_1)}_{2r-1}\zeta^{(i_2)}_0 - \zeta^{(i_1)}_0\zeta^{(i_2)}_{2r-1}\right)\Biggr) + \frac{1}{\pi}\sqrt{2\alpha_q}\left(\xi^{(i_1)}_q\zeta^{(i_2)}_0 - \zeta^{(i_1)}_0\xi^{(i_2)}_q\right) - \mathbf{1}_{\{i_1=i_2\}}\Biggr). \tag{12}$$

The approximation $I^{(i_3 i_2 i_1)q}_{(111)T,t}$ includes $\xi^{(i_1)}_q$ and $\xi^{(i_3)}_q$ $\Rightarrow$ the number q must be the same for all approximations $I^{(i_1 0)q}_{(10)T,t}$, $I^{(0 i_1)q}_{(01)T,t}$, $I^{(i_2 i_1)q}_{(11)T,t}$, $I^{(i_3 i_2 i_1)q}_{(111)T,t}$ $\Rightarrow$ huge computational costs due to complex approximation $I^{(i_3 i_2 i_1)q}_{(111)T,t}$ (P.E. Kloeden, and E. Platen (1992)).

Moreover, for numerical modeling of Gaussian random variables $I^{(i_1 0)}_{(10)T,t}$, $I^{(0 i_1)}_{(01)T,t}$ we need a large number q of $N_{i.i.d.}(0,1)$ random variables (see (10), (11)). For the case of Legendre polynomials we need only two $N_{i.i.d.}(0,1)$ random variables (see (8), (9)).

If we exclude $\xi_q^{(i_1)}$, $\xi_q^{(i_2)}$, $\xi_q^{(i_3)}$ from approximations $I_{(10)T,t}^{(i_1 0)q}$, $I_{(01)T,t}^{(0 i_1)q}$, $I_{(11)T,t}^{(i_2 i_1)q}$, $I_{(111)T,t}^{(i_3 i_2 i_1)q}$, then numbers q can be choosen different for approximations $I_{(10)T,t}^{(i_1 0)q}$, $I_{(01)T,t}^{(0 i_1)q}$, $I_{(11)T,t}^{(i_2 i_1)q}$, $I_{(111)T,t}^{(i_3 i_2 i_1)q}$. However:

$$\mathbf{E}\left(I_{(01)T,t}^{(0i_1)} - I_{(01)T,t}^{(0i_1)q}\right)^2 = \mathbf{E}\left(I_{(10)T,t}^{(i_2 0)} - I_{(10)T,t}^{(i_2 0)q}\right)^2 =$$

$$= \frac{(T-t)^3}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right) \neq 0 \quad (= 0 \text{ if with } \xi_q^{(i_1)}, \xi_q^{(i_2)}, \xi_q^{(i_3)}),$$

$$\mathbf{E}\left(I_{(11)T,t}^{(i_2 i_1)} - I_{(11)T,t}^{(i_2 i_1)q}\right)^2 = \frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right)$$

("1" instead of "3" if with $\xi_q^{(i_1)}$, $\xi_q^{(i_2)}$, $\xi_q^{(i_3)}$).

5. Comparison with Milstein method of mean-square approximation of iterated stochastic integrals

G.N. Milstein, "Numerical Integration of Stochastic Differential Equations" (In Russian), Sverdlovsk: Ural University Publ., 1988

P.E. Kloeden, E. Platen and I. Wright. "The approximation of multiple stochastic integrals", *Stoch. Anal. Appl.* **10**:4, 1992, 431-441

P.E. Kloeden and E. Platen, "Numerical Solution of Stochastic Differential Equations", Berlin: Springer, 1992 (1995 2nd Ed., 1999 3rd Ed.)

E. Platen and N. Bruti-Liberati, "Numerical Solution of Stochastic Differential Equations With Jumps in Finance", Berlin–Heidelberg: Springer, 2010.

The Milstein Method (G.N. Milstein (1988)).

Consider the Brownian bridge process

$$\left\{\mathbf{w}_t^{(i)} - \frac{t}{\Delta}\mathbf{w}_\Delta^{(i)},\ t \in [0, \Delta]\right\},\ \Delta > 0;\ i = 1, \ldots, m.$$

The Karhunen-Loeve expansion for this process has the form

$$\mathbf{w}_t^{(i)} - \frac{t}{\Delta}\mathbf{w}_\Delta^{(i)} = \frac{1}{2}a_{i,0} + \underset{q\to\infty}{\text{l.i.m.}} \sum_{r=1}^{q}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \quad (13)$$

$$a_{i,r} = \frac{2}{\Delta}\int_0^\Delta \left(\mathbf{w}_s^{(i)} - \frac{s}{\Delta}\mathbf{w}_\Delta^{(i)}\right)\ \cos\frac{2\pi rs}{\Delta}ds;\ r = 0,\ 1, \ldots,$$

$$b_{i,r} = \frac{2}{\Delta}\int_0^\Delta \left(\mathbf{w}_s^{(i)} - \frac{s}{\Delta}\mathbf{w}_\Delta^{(i)}\right)\ \sin\frac{2\pi rs}{\Delta}ds;\ r = 0,\ 1, \ldots.$$

Not difficult to show that random variables $a_{i,r},\ b_{i,r}$ are Gaussian, and for $i, i_1, i_2 = 1, \ldots, m;\ r \neq k;\ i_1 \neq i_2$:

$$\mathsf{M}\left\{a_{i,r}b_{i,r}\right\} = \mathsf{M}\left\{a_{i,r}b_{i,k}\right\} = \mathsf{M}\left\{a_{i,r}a_{i,k}\right\} = \mathsf{M}\left\{b_{i,r}b_{i,k}\right\} =$$

$$= \mathsf{M}\left\{a_{i_1,r}a_{i_2,r}\right\} = \mathsf{M}\left\{b_{i_1,r}b_{i_2,r}\right\} = 0,$$

$$\mathsf{M}\left\{a_{i,r}^2\right\} = \mathsf{M}\left\{b_{i,r}^2\right\} = \frac{\Delta}{2\pi^2 r^2}.$$

G.N. Milstein (1988): $\mathbf{w}_t^{(i)} \approx \mathbf{w}_t^{(i)n}$, and

$$\int_t^T \int_t^s d\mathbf{w}_\tau^{(i_1)} d\mathbf{w}_s^{(i_2)} \approx \int_t^T \int_t^s d\mathbf{w}_\tau^{(i_1)n} d\mathbf{w}_s^{(i_2)}\ \ (i_1 \neq i_2)\ \ \Rightarrow (12),$$

where

$$\mathbf{w}_t^{(i)n} = \mathbf{w}_\Delta^{(i)} \frac{t}{\Delta} + \frac{1}{2}a_{i,0} + \sum_{r=1}^{n}\left(a_{i,r}\cos\frac{2\pi r t}{\Delta} + b_{i,r}\sin\frac{2\pi r t}{\Delta}\right).$$

P.E. Kloeden, E. Platen and I. Wright (1992):

$$\int_t^T \int_t^{t_3} \int_t^{t_2} \circ\, d\mathbf{w}_{t_1}^{(i_1)} \circ d\mathbf{w}_{t_2}^{(i_2)} \circ d\mathbf{w}_{t_3}^{(i_3)} \approx$$

$$\approx \int_t^T \int_t^{t_3} \int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)n_1} d\mathbf{w}_{t_2}^{(i_2)n_2} d\mathbf{w}_{t_3}^{(i_3)n_3} \qquad (14)$$

where

$$\mathbf{w}_t^{(i)n} = \mathbf{w}_{\Delta}^{(i)} \frac{t}{\Delta} + \frac{1}{2} a_{i,0} + \sum_{r=1}^{n} \left(a_{i,r} \cos \frac{2\pi r t}{\Delta} + b_{i,r} \sin \frac{2\pi r t}{\Delta} \right).$$

Obviously (14) leads to iterated operation of limit transition. From particular case of the theorem 5 [1-4] (the case (3), $\phi_j(x)$ is $H_{t,T}$) we obtain the expansion in the right-hand side of (14).

Advantages of the method, which is based on the theorem 1.

(1) There is explicit formula for calculation of expansion coefficients $C_{j_k \ldots j_1}$ of iterated stochastic integrals (see theorem 1);

(2) There exists a possibility for exact calculation or effective estimation of mean-square error of approximation (see theorems 2 – 4);

(3) The basis functions is not only a trigonometric functions;

(4) The theorem 1 leads to only one operation of limit transition

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k}$$

(for example, $p_1 = \ldots = p_k = p \to \infty$). At the same time the Milstein method leads to iterated operation of limit transition

$$\lim_{p_1\to\infty} \sum_{j_1=0}^{p_1} \ldots \lim_{p_k\to\infty} \sum_{j_k=0}^{p_k}$$

starting at least from 2nd or 3rd multiplicity of iterated stochastic integral.

Thanks for your attention!