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BOOK of Abstracts

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Application of the Fourier Method for the Numerical Solution of Stochastic Differential Equations

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Abstract. It is well known, that Ito stochastic differential equations (SDEs) are adequate mathematical models of dynamic systems under the influence of random disturbances. One of the effective approaches to numerical integration of Ito SDEs is an approach based on Taylor-Ito and Taylor-Stratonovich expansions. The most important feature of such expansions is a presence in them of so called iterated Ito or Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs. We successfully use the tool of generalized multiple Fourier series, built in the space L_2, for the mean-square approximation of iterated stochastic integrals.

Keywords. iterated Ito stochastic integral; iterated Stratonovich stochastic inegral; Taylor-Ito expansion; strong approximation; numerical modeling.

MSC2010. 60H10; 60H05; 60H35; 65C30.

1 Introduction

Let $(\Omega, \mathrm{F}, \mathsf{P})$ be a fixed probability space and $\mathbf{W}_t$ — is F_t-measurable $\forall t \in [0,T]$ Wiener process with independent components $\mathbf{W}_t^{(i)};\ i = 1, \ldots, m$. Consider an Ito SDE:

$$d\mathbf{X}_t = \mathbf{a}(\mathbf{X}_t, t)dt + B(\mathbf{X}_t, t)d\mathbf{W}_t,\ \mathbf{X}_0 = \mathbf{X}(0,\omega),\ \omega \in \Omega, \tag{1}$$

where $\mathbf{a} : \Re^n \times [0,T] \to \Re^n$, $B : \Re^n \times [0,T] \to \Re^{n\times m}$ satisfy to standard conditions of existence and uniqueness of strong solution $\mathbf{X}_t \in \Re^n$ of the SDE (1); $\mathbf{X}_0$ and $\mathbf{W}_t - \mathbf{W}_0$ $(t > 0)$ — are independent. In theorems $1 - 3$ we solve the problem of combined mean-square approximation of stochastic integrals from Taylor-Ito and Taylor-Stratonovich expansions for the prosess $\mathbf{X}_t$.

2 Main results

Theorem 1 [1, p. 252]. *Assume, that* $\psi_i(\tau) \in C_{[t,T]}$ $(i = 1,\ 2, \ldots, k)$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *— is a complete orthonormal system of continuous functions in* $L_2([t,T])$. *Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \prod_{s=1}^{k} \phi_{j_s}(\tau_{l_s})\Delta\mathbf{W}_{\tau_{l_s}}^{(i_s)}\right),$$

where $J[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \ldots \int_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)} \ldots d\mathbf{W}_{t_k}^{(i_k)}$ *(iterated Ito stochastic integral),* $\Delta\mathbf{W}_{\tau_j}^{(i)} = \mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}$ $(i = 0,\ 1,\ \ldots, m)$, $\mathbf{W}_\tau^{(0)} = \tau$, $\zeta_j^{(i)} = \int_t^T \phi_j(\tau)d\mathbf{W}_\tau^{(i)}$ *— are independent* *standard Gaussian random variables for various* i *or* j *(if* $i \neq 0$), $\{\tau_j\}_{j=0}^{N-1}$ *— is a partition of* $[t,T]$, *satisfying to conditions*: $t=\tau_0<\ldots<\tau_N=T$, $\max_{0\le j\le N-1}(\tau_{j+1}-\tau_j)\to 0$ *if* $N\to\infty$, $C_{j_k\ldots j_1}=\int_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^k\phi_{j_l}(t_l)dt_1\ldots dt_k$, $K(t_1,\ldots,t_k)=\mathbf{1}_{\{t_1<\ldots<t_k\}}\psi_1(t_1)\ldots\psi_k(t_k)$ $(t_1,\ldots,t_k\in[t,T])$, $\mathbf{1}_A$ *— is an indicator of the set* A, $\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k$, $\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\neq l_r\ (g\neq r);\ g,r=1,\ldots,k\}$, $\mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\}$, l.i.m. *— is a limit in the mean-square sense.*

Consider particular cases of the theorem 1 for $k=2,3,4$ [1, pp. 261-262]:

$$J[\psi^{(2)}]_{T,t}=\sum_{j_1,j_2=0}^{\infty}C_{j_2j_1}(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\neq 0,j_1=j_2\}}),$$

$$J[\psi^{(3)}]_{T,t}=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\neq 0,j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_2=i_3\neq 0,j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\neq 0,j_1=j_3\}}\zeta_{j_2}^{(i_2)}),$$

$$J[\psi^{(4)}]_{T,t}=\sum_{j_1,\ldots,j_4=0}^{\infty}C_{j_4\ldots j_1}(\prod_{l=1}^4\zeta_{j_l}^{(i_l)}-\mathbf{1}_{\{i_1=i_2\neq 0,j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\neq 0,j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$
$$-\mathbf{1}_{\{i_1=i_4\neq 0,j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\neq 0,j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_2=i_4\neq 0,j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-$$
$$-\mathbf{1}_{\{i_3=i_4\neq 0,j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_2\neq 0,j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0,j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\neq 0,j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0,j_2=j_4\}}+$$
$$+\mathbf{1}_{\{i_1=i_4\neq 0,j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0,j_2=j_3\}}).$$

Consider some estimates for the mean-square error of approximation, based on the theorem 1.

Theorem 2 [1, pp. 500-501]. *Under conditions of the theorem* 1:

$$\mathsf{M}\{(J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}-J[\psi^{(k)}]_{T,t})^2\}\le k!(\int_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{p_1,\ldots,p_k}C_{j_k\ldots j_1}^2)$$
$$(i_1,\ldots,i_k=0,1,\ldots,m\ \textit{and}\ 0<T-t<1\ \textit{or}\ i_1,\ldots,i_k=1,\ldots,m\ \textit{and}\ 0<T-t<\infty)$$
$$\textit{and}$$
$$\mathsf{M}\{(J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}-J[\psi^{(k)}]_{T,t})^2\}=\int_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{p_1,\ldots,p_k}C_{j_k\ldots j_1}^2$$
$$(i_1,\ldots,i_k=1,\ldots,m\ \textit{and pairwise different};\ 0<T-t<\infty\ ),$$

where $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ *is a truncated series from the theorem* 1 *with upper limits* $p_1,\ldots,p_k$, *and* M *— is a mathematical expectation.*

The following theorem adapts the theorem 1 for iterated Stratonovich stochastic integrals.

Theorem 3 [1, pp. 284-428]. *Let function* $\psi_2(s)$ *— is continuously differentiated at* $[t,T]$ *and functions* $\psi_1(s)$, $\psi_3(s)$ *— are two times continuously differentiated at* $[t,T]$; $\{\phi_j(x)\}_{j=0}^{\infty}$ *— is a complete orthonormal system of Legendre polynomials or trigonometric functions in* $L_2([t,T])$. *Then*

$$J^*[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},\ \ J^*[\psi^{(k)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)},$$

where $J^*[\psi^{(k)}]_{T,t}=\int_t^{*T}\psi_k(t_k)\ldots\int_t^{*t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}$ *(iterated Stratonovich stochastic integral)*; $k=3,4,5$ *(for* $k=3: i_1,i_2,i_3=1,\ldots,m$; *for* $k=4,5: i_1,\ldots,i_5=0,1,\ldots,m$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)\equiv 1$); *the meaning of notations from the theorem* 1 *is remained.*

References

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