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Latest results on a new approach to series expansion of iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process. Multiplicities 1 to 8 and beyond

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1 Introduction

The importance of the problem of numerical integration of SDEs is explained by a wide range of their applications related to the construction of adequate mathematical models of dynamic systems under random disturbances and to the application of SDEs for solving various mathematical problems, among which we mention signal filtering, stochastic optimal control, stochastic stability, evaluating the parameters of stochastic systems.

Iterated Itô and Stratonovich stochastic integrals can be used to construct high-order strong (mean-square) numerical methods for various types of systems of SDEs with non-commutative noise. For example, for

- Itô stochastic differential equations
- Itô stochastic differential equations with jumps
- McKean stochastic differential equations
- stochastic differential equations with switchings
- semilinear stochastic partial differential equations with multiplicative trace class noise

Let $(\Omega,\ F,\ \mathsf{P})$ be a complete probability space, let $\{F_t, t\in[0,\ T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{W}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t\in[0,\ T]$. We assume that the components $\mathbf{W}_t^{(i)}$ $(i=1,\ldots,m)$ of this process are independent. As an example, consider a system of Itô SDEs with non-commutative noise

$$\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\sum_{j=1}^m\int\limits_0^t B_j(\mathbf{x}_\tau,\tau)d\mathbf{W}_\tau^{(j)},\ \ \mathbf{x}_0=\mathbf{x}(0,\omega), \qquad (1)$$

where $\mathbf{x}_t\in\mathbf{R}^n$, the nonrandom functions $\mathbf{a},\ B_j:\mathbf{R}^n\times[0,\ T]\to\mathbf{R}^n$ guarantee the existence and uniqueness up to stochastic equivalence of a strong solution of (1), $\mathbf{x}_0$ is F_0-measurable, $\mathbf{E}\left|\mathbf{x}_0\right|^2<\infty$, $\mathbf{x}_0$ and $\mathbf{W}_t-\mathbf{W}_0$ are independent when $t>0$.

• Suppose that $\mathbf{a}$ and B_j $(j=1,\ldots,m)$ are several times continuously differentiable with respect to both arguments and noise is non-commutative, i.e. the following relations are not fulfulled

$$G_iB_j=G_jB_i, G_iG_jB_k=G_jG_iB_k,\ldots(i,j,k,\ldots=1,\ldots,m), G_i=\sum_{k=1}^n B_{ki}\frac{\partial}{\partial\mathbf{x}_k}$$

One of the effective approaches to the numerical integration of Itô SDEs is based on the Taylor–Itô and Taylor–Stratonovich expansions. These expansions contain iterated Itô and Stratonovich stochastic integrals:

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \qquad (2)$$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots \circ d\mathbf{W}_{t_k}^{(i_k)}, \qquad (3)$$

where $\mathbf{W}_{\tau}^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes, $\psi_1(\tau), \ldots, \psi_k(\tau) : [t, T] \to \mathbf{R}$, $\mathbf{W}_{\tau}^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$, $d\mathbf{W}_{\tau}^{(i)}$ and $\circ d\mathbf{W}_{\tau}^{(i)}$ denote Itô and Stratonovich differentials, respectively.

• A natural question arises: is it possible to construct a numerical scheme for Itô SDE that includes only increments of the Wiener processes but has a higher order of convergence than the Euler method? It is known that this is impossible for non-commutative noise and $m > 1$ ("Clark–Cameron paradox"). This explains the need to use iterated stochastic integrals for constructing high-order strong numerical methods for Itô SDEs.

2 Brief Review of Old Results on Expansion of iterated Itô and Stratonovich Stochastic Integrals

Let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \tag{4}$$

where $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}$, braces mean an unordered set, and parentheses mean an ordered set.

Further, we will consider sums of the form

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}) \\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1\ldots q_{k-2r}},$$

for all $r = 1, 2, \ldots, [k/2]$ and for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (4)), where $a_{g_1 g_2,\ldots,g_{2r-1} g_{2r},q_1\ldots q_{k-2r}} \in \mathbf{R}$.

Theorem 1 [Kuz1] (2006,2023). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$. *Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underline{\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr),$$

where $k\in\mathbf{N}$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{W}_\tau^{(i)}$$

are i.i.d. $N(0,1)$*-r.v.'s for various* i *or* j (*if* $i\neq 0$), $i_1,\ldots,i_k=0,1,\ldots,m$, $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by (2),* $\mathbf{W}_\tau^{(i)}$ $(i=1,\ldots,m)$ *are independent standard Wiener processes,* $C_{j_k\ldots j_1}$ *is the Fourier coefficient for* $K(t_1,\ldots,t_k)=\psi_1(t_1)\ldots\psi_k(t_k)\mathbf{1}_{\{t_1<\ldots<t_k\}}$ $(k\geq 2)$ *and* $K(t_1)=\psi_1(t_1)$ $(k=1)$, $[x]$ *is an integer part of* x, $\mathbf{1}_A$ *is the indicator of* A, $\mathbf{W}_\tau^{(0)}=\tau$, $\prod_{\emptyset}\overset{\text{def}}{=}1$, $\sum_{\emptyset}\overset{\text{def}}{=}0$.

• **Remark.** The case $\psi_1(\tau),\ldots,\psi_k(\tau) \in C[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a CONS in $L_2[t,T]$ such that $\phi_j(x) \in C[t,T]$ or $\phi_j(x)$ is piecewise continuous on $[t,T]$ $\forall j \in \mathbf{N}$ has been considered in **[Kuz1]** (2006–2009).

• **Remark.** Another form (based on explicit product of Hermite polynomials) of the expansion from Theorem 1 can be found in **[Ryb1]** (2021).

Let us consider particular cases of Theorem 1 for $k=1,\ldots,4$

$$J[\psi^{(1)}]_{T,t}^{(i_1)} = \underset{p_1\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} C_{j_1}\zeta_{j_1}^{(i_1)},$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right),$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr),$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

where $\mathbf{1}_A$ is the indicator of the set A.

Recall that

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \qquad (5)$$

is the Fourier coefficient corresponding to the factorized Volterra-type kernel

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ 0, & \text{otherwise}\end{cases}\quad (k\ge 2), \qquad (6)$$

and $K(t_1)=\psi_1(t_1)$ $(k=1)$, where $t_1,\ldots,t_k\in[t, T]$.

Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be an arbitrary CONS in $L_2[t, T]$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t, T]$.

Denote

$$C_{j_k\ldots j_{l+1}\ \underline{j_l\ j_{l-1}}\ j_{l-2}\ldots j_1}\Bigg|_{(j_l j_{l-1})\curvearrowright(\cdot)} \stackrel{\rm def}{=}$$

$$\stackrel{\rm def}{=} \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int\limits_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int\limits_t^{t_{l+1}} \underline{\psi_l(t_l)\psi_{l-1}(t_l)}\times$$

$$\times \int\limits_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}\ \underline{dt_l}\ t_{l+1}\ldots dt_k,$$

where we suppose that $\{l, l-1\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$ (see (4)).

Denote

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_l,\ldots,s_1]} \stackrel{\rm def}{=} \prod_{p=1}^{l} \mathbf{1}_{\{i_{s_p}=i_{s_p+1}\neq 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \underline{\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})}\times$$

$$\times \int\limits_t^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1})\ldots \int\limits_t^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_t^{t_{s_1+2}} \underline{\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})}\times$$

$$\times \int\limits_t^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1})\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_{s_1-1}}^{(i_{s_1-1})}\ \underline{dt_{s_1+1}}\ d\mathbf{W}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{W}_{t_{s_l-1}}^{(i_{s_l-1})}\ \underline{dt_{s_l+1}}\ d\mathbf{W}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{W}_{t_k}^{(i_k)}, \qquad (7)$$

where $(s_l, \ldots, s_1) \in A_{k,l}$,

$$A_{k,l} = \left\{(s_l, \ldots, s_1) : s_l > s_{l-1}+1, \ldots, s_2 > s_1+1; s_l, \ldots, s_1 = 1, \ldots, k-1\right\}, \tag{8}$$

$l = 1, 2, \ldots, [k/2]$, $i_1, \ldots, i_k = 0, 1, \ldots, m$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

Denote

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \stackrel{\text{def}}{=} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in A_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}. \tag{9}$$

Let us formulate the statement on connection between iterated Itô and Stratonovich stochastic integrals of arbitrary multiplicity k.

Theorem 2 [Kuz1] (1997). *Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$. Then, the following relation between iterated Stratonovich and Itô stochastic integrals is correct*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \quad \text{w. p. 1},$$

where $i_1, \ldots, i_k = 0, 1, \ldots, m$, $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Theorem 3 [Kuz2] (2024). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$ *and the following condition*

$$\lim_{p\to\infty}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\right.$$

$$\left.-\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}\right)^2=0 \qquad (10)$$

is satisfied for all $r=1,2,\ldots,[k/2]$ *and for all possible* $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see (4)). Then*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}. \qquad (11)$$

If in addition $\psi_1(\tau),\ldots,\psi_k(\tau)\in C[t,T]$, then

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}.$$

Condition (10) has been verified in several cases (see below).

For Special CONS in $L_2[t, T]$

Theorem 4 **[Kuz1]** (2018, 2022). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric Fourier basis in* $L_2[t, T]$ *and* $\psi_1(\tau), \ldots, \psi_5(\tau) \in C^1[t, T]$. *Then*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)} \quad (k = 2, 3, 4, 5), \quad (12)$$

$$\mathbf{E}\left(J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)}\right)^2 \le \frac{C}{p}\mathbf{1}_{\{k=2,3\}} + \frac{C}{p^{1-\varepsilon}}\mathbf{1}_{\{k=4,5\}}, \quad (13)$$

where $i_1, \ldots, i_5 = 0, 1, \ldots, m$ *in (12) and* $i_1, \ldots, i_5 = 1, \ldots, m$ *in (13),* ε *is an arbitrary small positive real number for the polynomial case and* $\varepsilon = 0$ *for the trigonometric case, constant* C *is independent of* p*; another notations as in Theorem 1.*

Theorem 5 **[Kuz1]** (2022). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric Fourier basis in* $L_2[t,T]$ *and* $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$. *Then*

$$J^*[\psi^{(6)}]_{T,t}^{(i_1\ldots i_6)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)},$$

where $i_1,\ldots,i_6 = 0,1,\ldots,m$; *another notations as in Theorem 1.*

For an Arbitrary CONS in $L_2[t,T]$

Theorem 6 **[Kuz1]** (2024). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$, $\psi_1(\tau),\psi_2(\tau)\in C[t,T]$ $(k=2)$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)\equiv 1$ $(k=3,4,5)$. *Then*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \quad (k=2,3,4,5),$$

where $i_1,\ldots,i_5 = 0,1,\ldots,m$; *another notations as in Theorem 1.*

3 New Results on Expansion of Iterated Stratonovich Stochastic Integrals.

3.1 Calculation of Matrix Traces of Volterra-Type Integral Operators

Recall the condition (10) from Theorem 3 for $k \geq 2r$

$$\lim_{p\to\infty} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)} \right)^2 = 0$$

and for $k = 2r$

$$\lim_{p\to\infty} \sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}. \qquad (14)$$

As a result, we must at least prove the equality (14), since (14) is a particular case of (10) for $k = 2r$.

It is easy to see that the following factorized Volterra-type kernel

$$K(t_1, \ldots, t_k) = \psi_1(t_1) \ldots \psi_k(t_k) \mathbf{1}_{\{t_1 < \ldots < t_k\}} \ \ (k \geq 2) \tag{15}$$

for even $k = 2r$ ($r \in \mathbf{N}$) forms a family of integral operators $\mathbb{K} : L_2([t, T]^r) \to L_2([t, T]^r)$ of the form

$$(\mathbb{K}f)(t_{q_1}, \ldots, t_{q_r}) = \int\limits_{[t,T]^r} K(t_1, \ldots, t_k) f(t_{q_{r+1}}, \ldots, t_{q_k}) dt_{q_{r+1}} \ldots dt_{q_k}, \tag{16}$$

where $\{q_1, \ldots, q_k\} = \{1, \ldots, k\}$, $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2[t, T]$, $t_1, \ldots, t_k \in [t, T]$ $(k \geq 2)$.

The equality (14) is the equality between matrix and integral traces of the integral operator (16).

For example,

$$\left(\mathbb{K}f\right)(t_3,t_4)=\int\limits_{[t,T]^2} K(t_1,\ldots,t_4)f(t_1,t_2)dt_1dt_2=$$

$$=\mathbf{1}_{\{t_3<t_4\}}\psi_3(t_3)\psi_4(t_4)\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)f(t_1,t_2)dt_1dt_2.$$

• **Remark.** It is well known that the Volterra integral operator (the simplest operator from the family (16)) is not a trace class operator. On the other hand, it is known that for trace class operators the equality of matrix and integral traces holds. It is known that for the Volterra integral operator (although it is not a trace class operator), the equality of matrix and integral traces is also true. Thus, one cannot count on the fact that operators of the more general form (16) are operators of the trace class.

As a result, the proof of the equalities of matrix and integral traces for Volterra–type integral operators (16) is a problem.

Theorem 7. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$. *Then the equality*

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)} \qquad (17)$$

is satisfied for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ *(see (4)) and for any fixed* $j_{q_1},\ldots,j_{q_{k-2r}}$ $(\{q_1,\ldots,q_{k-2r}\}=\{1,\ldots,k\}\backslash\{g_1,g_2,\ldots,g_{2r-1},g_{2r}\})$, *where* $k\ge 2r$ *and* $r=1,2,\ldots,[k/2]$. *Furthermore, the series (17) converges absolutly for* $k=2r$ *and converges absolutely for any fixed* $q_1,\ldots,q_{k-2r}$ *for* $k>2r$.

- **Remark.** From Theorem 7 it follows that

$$\lim_{p\to\infty}\mathbf{E}\left(J^*[\psi^{(k)}]_{T,t}-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}\right)=0.$$

Outline of the Proof of Theorem 7

Step 1. The case $k = 2$ and $r = 1$. The equality (17) for the case $k = 2$ and $r = 1$ looks as follows

$$\sum_{j=0}^{\infty} \int_t^T \psi_2(t_2)\phi_j(t_2) \int_t^{t_2} \psi_1(\tau)\phi_j(t_1)dt_1dt_2 = \frac{1}{2}\int_t^T \psi_1(\tau)\psi_2(\tau)d\tau, \qquad (18)$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t, T]$ and $\psi_1(\tau), \psi_2(\tau) \in L_2[t, T]$.

The equality (18) is proved in

- **[Kuz1]** (2011) for special CONS in $L_2[t, T]$ and $\psi_1(\tau), \psi_2(\tau) \in C^1[t, T]$.
- **[Ryb2]** (2023) for an arbitrary CONS in $L_2[t, T]$ and $\psi_1(\tau), \psi_2(\tau) \in L_2[t, T]$.
- **[Kuz1]** (2023) for an arbitrary CONS in $L_2[t, T]$ and $\psi_1(\tau) = (\tau - t)^m$, $\psi_2(\tau) = (\tau - t)^n$; $m, n = 0, 1, 2, \ldots$

Step 2. The case $k=2r$ of (17) for

$$\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}=1,$$

i.e. all pairs are formed by adjacent indices

$$\lim_{p\to\infty}\sum_{j_{2r},j_{2r-2},\ldots,j_2=0}^{p} C_{j_{2r}j_{2r}j_{2r-2}j_{2r-2}\ldots j_2j_2}=\frac{1}{2^r}\int_t^T\psi_{2r}(t_{2r})\psi_{2r-1}(t_{2r})\times$$

$$\times\int_t^{t_{2r}}\psi_{2r-2}(t_{2r-2})\psi_{2r-3}(t_{2r-2})\ldots\int_t^{t_4}\psi_2(t_2)\psi_1(t_2)dt_2\ldots dt_{2r-2}dt_{2r}, \quad (19)$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$, $\psi_1(\tau),\ldots,\psi_{2r}(\tau)\in L_2[t,T]$ and $r\in\mathbf{N}$.

- **[Ryb2]** (2023) (the proof of (19) on the base of trace class operators).
- **[Kuz2]** (2024) (the proof of (19) on the base of Step 1 and induction).

Step 3. The case $k = 2r$ of (17) for

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 0.$$

This case is considered in:

- **[Ryb2]** (2024) (the proof on the base of the theory of trace class operators).
- **[Kuz2]** (2024) (the proof of on the base of Parseval's equality and Step 2).

An example on the proof of Step 3 will be given in Appendix 1.

Step 4. General case $k \geq 2r$ of (17).

- **[Kuz2]** (2024).

This proof of Step 4 will be given in Appendix 2.

Using Theorem 7, consider some sufficient conditions under which Theorem 3 is true.

Suppose that

$$\exists \quad A \stackrel{\rm def}{=} \lim_{p,q\to\infty} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{q} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}\right)^2 < \infty \quad (20)$$

for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (4)) and for all $r = 1, 2, \ldots, [k/2]$. Then by Theorem 7

$$A = \lim_{q\to\infty} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{q} \lim_{p\to\infty} \left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}\right)^2 = 0.$$

Substituting $p = q$ into (20), we obtain (10) and Theorem 3 is true. Thus, we have the following theorem.

Theorem 8 [Kuz2] (2024). *Suppose that the condition (20) is fulfilled, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t, T]$, $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2[t, T]$. Then, for the sum of iterated Itô stochastic integrals*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} \stackrel{\rm def}{=} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in A_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$$

the following expansion

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\rm l.i.m.} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

holds. If in addition $\psi_1(\tau), \ldots, \psi_k(\tau) \in C[t, T]$, then for

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) \circ d\mathbf{W}_{t_1}^{(i_1)} \ldots \circ d\mathbf{W}_{t_k}^{(i_k)}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\rm l.i.m.} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}$$

holds, where notations are the same as in Theorem 1.

3.2 New Approach Based on Parseval's Equality and Dominated Convergence Theorem

We will start this section with an example. Let $h_1(\tau), \ldots, h_{12}(\tau) \in L_2([t, T])$ and consider the following integral

$$I \stackrel{\rm def}{=} \int\limits_t^T h_{12}(t_{12}) \int\limits_t^{t_{12}} h_{11}(t_{11}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{11} dt_{12}.$$

We want to transform the integral I in such a way that

$$I = \int\limits_t^T h_{10}(t_{10}) \int\limits_t^{t_{10}} h_6(t_6) \int\limits_t^{t_6} h_4(t_4) \int\limits_t^{t_4} h_3(t_3) (\ldots)\, dt_3 dt_4 dt_6 dt_{10},$$

where $(\ldots)$ is some expression.

Using Fubini's Theorem, we obtain

$$I=\int_t^T h_{12}(t_{12})\int_t^{t_{12}} h_{11}(t_{11})\int_t^{t_{11}} h_{10}(t_{10})\int_t^{t_{10}} h_9(t_9)\int_t^{t_9} h_8(t_8)\int_t^{t_8} h_7(t_7)\int_t^{t_7} h_6(t_6)\times$$

$$\times\int_t^{t_6} h_5(t_5)\int_t^{t_5} h_4(t_4)\int_t^{t_4} h_3(t_3)\int_t^{t_3} h_2(t_2)\int_t^{t_2} h_1(t_1)dt_1dt_2dt_3dt_4dt_5dt_6dt_7dt_8\times$$

$$\times dt_9dt_{10}dt_{11}dt_{12}=\ldots=$$

$$=\int_t^T h_{10}(t_{10})\int_t^{t_{10}} h_6(t_6)\int_t^{t_6} h_4(t_4)\int_t^{t_4} h_3(t_3)\left(\int_t^{t_3} h_2(t_2)\int_t^{t_2} h_1(t_1)dt_1dt_2\right)\times$$

$$\times\left(\int_{t_4}^{t_6} h_5(t_5)dt_5\right)\left(\int_{t_6}^{t_{10}} h_9(t_9)\int_{t_6}^{t_9} h_8(t_8)\int_{t_6}^{t_8} h_7(t_7)dt_7dt_8dt_9\right)\times$$

$$\times\left(\int_{t_{10}}^{T} h_{12}(t_{12})\int_{t_{10}}^{t_{12}} h_{11}(t_{11})dt_{11}dt_{12}\right)dt_3dt_4dt_6dt_{10}. \qquad (21)$$

Denote

$$C_{j_k\ldots j_1}^{\psi_k\ldots\psi_1}(s,\tau)=\int\limits_{\tau}^{s}\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_{\tau}^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k,$$

where $t \le \tau < s \le T$, and suppose that $h_l(\tau)=\psi_l(\tau)\phi_{j_l}(\tau)$ ($l = 1,\ldots,12$) in (21) (here $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_{12}(\tau)\in L_2([t,T])$). Then

$$C_{j_{12}j_{11}j_{10}j_9j_8j_7j_6j_5j_4j_3j_2j_1}=\int\limits_{t}^{T}\psi_{10}(t_{10})\phi_{j_{10}}(t_{10})\int\limits_{t}^{t_{10}}\psi_6(t_6)\phi_{j_6}(t_6)\int\limits_{t}^{t_6}\psi_4(t_4)\phi_{j_4}(t_4)\times$$

$$\times\int\limits_{t}^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)C_{j_{12}j_{11}}^{\psi_{12}\psi_{11}}(T,t_{10})C_{j_9j_8j_7}^{\psi_9\psi_8\psi_7}(t_{10},t_6)C_{j_5}^{\psi_5}(t_6,t_4)C_{j_2j_1}^{\psi_2\psi_1}(t_3,t)\times$$

$$\times\, dt_3dt_4dt_6dt_{10}. \qquad (22)$$

Suppose that $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (4) and $k > 2r$, $r \geq 1$ (the case $k = 2r$ is proved in Theorem 7).

Consider $d_1, e_1, \ldots, d_f, e_f, f \in \mathbf{N}$ such that

$$1 \leq d_1 - e_1 + 1 < \ldots < d_1 - 1 < d_1 < \ldots < d_f - e_f + 1 < \ldots < d_f - 1 < d_f \leq k,$$

$$\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\} =$$

$$= \{d_1 - e_1 + 1, \ldots, d_1 - 1, d_1\} \cup \ldots \cup \{d_f - e_f + 1, \ldots, d_f - 1, d_f\},$$

$$e_1 + e_2 + \ldots + e_f = 2r, \ \ \{1, \ldots, k\} \setminus \{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\} = \{q_1, \ldots, q_{k-2r}\}.$$

• *We will say that the condition* **(A)** *is satisfied if* $\forall \ \{g_l, g_{l+1}\}$ $(l = 1, \ldots, 2r-1)$ $\exists \ h \in \{1, \ldots, f\}$ *such that*

$$\{g_l, g_{l+1}\} \subset \{d_h - e_h + 1, \ldots, d_h - 1, d_h\}. \qquad (23)$$

Moreover, $\forall \ h \in \{1, \ldots, f\}$ $\exists \ \{g_l, g_{l+1}\}$ $(l = 1, \ldots, 2r-1)$ *such that (23) is fulfilled.*

For definiteness, let $q_1 < \ldots < q_{k-2r}$, $k > 2r$, $r \geq 1$. Using Fubini's Theorem (as in the above example (see (21), (22)), we obtain

$$\sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} C_{j_k \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \int\limits_t^T \psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}}) \ldots \int\limits_t^{t_{q_1+1}} \psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1}) \times$$

$$\times \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} \Bigg(C_{j_{d_f} \cdots j_{d_f-e_f+1}}^{\psi_{d_f} \cdots \psi_{d_f-e_f+1}}(t_{d_f+1}, t_{d_f-e_f}) \ldots$$

$$\ldots C_{j_{d_1} \cdots j_{d_1-e_1+1}}^{\psi_{d_1} \cdots \psi_{d_1-e_1+1}}(t_{d_1+1}, t_{d_1-e_1}) \Bigg) \Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} \times$$

$$\times \, dt_{q_1} \ldots dt_{q_{k-2r}}, \qquad (24)$$

$$\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}=$$

$$=\int\limits_t^T\psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}})\ldots\int\limits_t^{t_{q_1+1}}\psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1})\times$$

$$\times\mathbf{1}_{\{the\ \ condition\ \ (A)\ \ is\ \ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times$$

$$\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Bigg|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot)}\times$$

$$\times\, dt_{q_1}\ldots dt_{q_{k-2r}}, \qquad (25)$$

where $t_{k+1}\stackrel{\rm def}{=}T,\ t_0\stackrel{\rm def}{=}t,\ e_1+\ldots+e_f=2r,\ r_1+\ldots+r_f=r.$

Lemma 1 [Kuz2] (2024). The following equality holds

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\left(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots\right.$$

$$\left.\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\right)\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}=$$

$$=\mathbf{1}_{\{the\ \ condition\ \ (A)\ \ is\ \ satisfied\}}\prod_{h=1}^{f}\frac{1}{2^{r_h}}\prod_{l=1}^{r_h}\mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}}\times$$

$$\times C_{j_{d_h}\ldots j_{d_h-e_h+1}}^{\psi_{d_h}\ldots\psi_{d_h-e_h+1}}(t_{d_h+1},t_{d_h-e_h})\Bigg|_{(j_{g_2^{(h)}}j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}}j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot)},\qquad(26)$$

where $t_{k+1}\stackrel{\rm def}{=}T$, $t_0\stackrel{\rm def}{=}t$, $e_1+\ldots+e_f=2r$, $r_1+\ldots+r_f=r$.

Proof. Using Theorem 7 and a modification of the proof of Theorem 7, we prove (26). Note that the series on the left-hand side of (26) converges absolutly since its sums does not depend on permutations of basis functions (here the basis in $L_2([t,T]^r)$ is $\{\phi_{j_1}(x_1)\ldots\phi_{j_r}(x_r)\}_{j_1,\ldots,j_r=0}^{\infty}$).

Suppose that

$$\left| \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} \left(C_{j_{d_f} \cdots j_{d_f - e_f + 1}}^{\psi_{d_f} \cdots \psi_{d_f - e_f + 1}} (t_{d_f+1}, t_{d_f - e_f}) \ldots \right. \right.$$

$$\left. \left. \ldots C_{j_{d_1} \cdots j_{d_1 - e_1 + 1}}^{\psi_{d_1} \cdots \psi_{d_1 - e_1 + 1}} (t_{d_1+1}, t_{d_1 - e_1}) \right) \right|_{j_{g_1} = j_{g_2}, \ldots, j_{g_{2r-1}} = j_{g_{2r}}} \Bigg| \le K < \infty, \qquad (27)$$

where constant K does not depend on p and $t_{d_1+1}, t_{d_1-e_1}, \ldots, t_{d_f+1}, t_{d_f-e_f}$ (here $d_1 - e_1 \ge 1$ and $d_f + 1 \le k$). In (27): $t_{k+1} \stackrel{\text{def}}{=} T$, $t_0 \stackrel{\text{def}}{=} t$, $e_1 + \ldots + e_f = 2r$; another notations as above in this section.

• **Remark.** The condition (27) can be weakened, i.e. the constant K^2 can be replaced by the function $F(t_{q_1}, \ldots, t_{q_{k-2r}})$ such that

$$\psi_{q_1}^2(t_{q_1}) \ldots \psi_{q_{k-2r}}^2(t_{q_{k-2r}}) F(t_{q_1}, \ldots, t_{q_{k-2r}}) \in L_1([t, T]^{k-2r})$$

(integrable majorant). Moreover, we can suppose that the weakened version of (27) is valid a.e. on $X = \{(t_{q_1}, \ldots, t_{q_{k-2r}}) : t \le t_{q_1} \le \ldots \le t_{q_{k-2r}} \le T\}$ with respect to Lebesgue's measure.

Applying (26), (24), (25), we obtain ($k > 2r,\ r \ge 1$)

$$\lim_{p\to\infty}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{p}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}\right)^2 \le$$

$$\le \lim_{p\to\infty}\sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{\infty}\left(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \right.$$

$$\left. -\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}\right)^2 =$$

$$
= \lim_{p\to\infty} \sum_{j_{q_1},\ldots,j_{q_{k-2r}}=0}^{\infty} \Biggl(\int\limits_t^T \psi_{q_{k-2r}}(t_{q_{k-2r}})\phi_{j_{q_{k-2r}}}(t_{q_{k-2r}}) \ldots \int\limits_t^{t_{q_1+1}} \psi_{q_1}(t_{q_1})\phi_{j_{q_1}}(t_{q_1}) \times
$$

$$
\times \Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p} \biggl(C_{j_{d_f}\cdots j_{d_f-e_f+1}}^{\psi_{d_f}\cdots\psi_{d_f-e_f+1}}(t_{d_f+1}, t_{d_f-e_f}) \ldots
$$

$$
\ldots C_{j_{d_1}\cdots j_{d_1-e_1+1}}^{\psi_{d_1}\cdots\psi_{d_1-e_1+1}}(t_{d_1+1}, t_{d_1-e_1}) \biggr) \Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} -
$$

$$
-\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}} \prod_{h=1}^{f} \frac{1}{2^{r_h}} \prod_{l=1}^{r_h} \mathbf{1}_{\{g_{2l}^{(h)}=g_{2l-1}^{(h)}+1\}} \times
$$

$$
\times C_{j_{d_h}\cdots j_{d_h-e_h+1}}^{\psi_{d_h}\cdots\psi_{d_h-e_h+1}}(t_{d_h+1}, t_{d_h-e_h}) \Biggr|_{(j_{g_2^{(h)}} j_{g_1^{(h)}})\curvearrowright(\cdot)\ldots(j_{g_{2r_h}^{(h)}} j_{g_{2r_h-1}^{(h)}})\curvearrowright(\cdot)} \Biggr) \times
$$

$$
\times dt_{q_1} \ldots dt_{q_{k-2r}} \Biggr)^2 = [Parseval's\ Eq.] = \qquad (28)
$$

$$
= \lim_{p\to\infty} \int\limits_t^T \psi^2_{q_{k-2r}}(t_{q_{k-2r}}) \ldots \int\limits_t^{t_{q_1+1}} \psi^2_{q_1}(t_{q_1}) \times
$$

$$
\times \Biggl(\sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} \Bigl(C^{\psi_{d_f} \cdots \psi_{d_f - e_f + 1}}_{j_{d_f} \cdots j_{d_f - e_f + 1}}(t_{d_f+1}, t_{d_f - e_f}) \ldots
$$

$$
\ldots C^{\psi_{d_1} \cdots \psi_{d_1 - e_1 + 1}}_{j_{d_1} \cdots j_{d_1 - e_1 + 1}}(t_{d_1+1}, t_{d_1 - e_1}) \Bigr) \Bigg|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} -
$$

$$
-\mathbf{1}_{\{the\ condition\ (A)\ is\ satisfied\}} \prod_{h=1}^{f} \frac{1}{2^{r_h}} \prod_{l=1}^{r_h} \mathbf{1}_{\{g^{(h)}_{2l} = g^{(h)}_{2l-1}+1\}} \times
$$

$$
\times C^{\psi_{d_h} \cdots \psi_{d_h - e_h + 1}}_{j_{d_h} \cdots j_{d_h - e_h + 1}}(t_{d_h+1}, t_{d_h - e_h}) \Bigg|_{(j_{g^{(h)}_2} j_{g^{(h)}_1}) \curvearrowright (\cdot) \ldots (j_{g^{(h)}_{2r_h}} j_{g^{(h)}_{2r_h - 1}}) \curvearrowright (\cdot),} \Biggr)^2 \times
$$

$$
\times dt_{q_1} \ldots dt_{q_{k-2r}} = [Dom.\ Conv.\ Th.\ and\ (26), (27)] = 0, \qquad (29)
$$

where the integrand function in (29) converges to zero (a.e. on $X = \{(t_{q_1}, \ldots, t_{q_{k-2r}})\colon t \le t_{q_1} \le \ldots \le t_{q_{k-2r}} \le T\}$ w.r. to Lebesgue's measure).

Thus, we have proved the following theorem.

Theorem 9 [Kuz2] (2024). *Suppose that the condition (27) is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2[t,T]$*. Then, for the sum* $\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of iterated Itô stochastic integrals (9) we have*

$$\bar{J}^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}.$$

If in addition $\psi_1(\tau),\ldots,\psi_k(\tau)\in C[t,T]$*, then for the iterated Stratonovich stochastic integral* $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *of multiplicity k (3) we have*

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \qquad (30)$$

where $i_1,\ldots,i_k=0,1,\ldots,m$*; another notations as in Theorem 1.*

Partial Proof of the Condition (27)

Lemma 2 **[Kuz1]** (2025). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2([t, T])$ *and* $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$. *Then*

$$\left|\sum_{j_r, j_{r-2}, \ldots, j_2=0}^{p} C_{j_r j_r j_{r-2} j_{r-2} \ldots j_2 j_2}(s, \tau)\right| \le K < \infty, \tag{31}$$

where $p \in \mathbf{N}$, $r = 2, 4, 6, \ldots,$ *constant* K *does not depend on* p, s, τ *(but only on* t, T*),*

$$C_{j_k \ldots j_1}(s, \tau) = \int_{\tau}^{s} \phi_{j_k}(t_k) \ldots \int_{\tau}^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_k,$$

where $k \in \mathbf{N}$, $t \le \tau < s \le T$.

• **Remark.** However, the proof of condition (27) in more general cases than (31) is the subject of the proof of the following three theorems.

On the base of Theorem 9, we proved the following 3 theorems.

Theorem 10 **[Kuz2]** (2024). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$. *Then, for the iterated Stratonovich stochastic integral*

$$I_{l_1\ldots l_k\,T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}(t_k-t)^{l_k}\ldots\int\limits_t^{*t_2}(t_1-t)^{l_1}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad (k=3,4)$$

the following expansion

$$I_{l_1\ldots l_k\,T,t}^{*(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\quad (k=3,4)$$

is valid, where $i_1,\ldots,i_4=0,1,\ldots,m;\ l_1,\ldots,l_4=0,1,2,\ldots,$

$$C_{j_k\ldots j_1}=\int\limits_t^{T}(t_k-t)^{l_k}\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}(t_1-t)^{l_1}\phi_{j_1}(t_1)dt_1\ldots dt_k\quad (k=3,4);$$

another notations as in Theorem 1.

Theorem 11 [Kuz1] (2025). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2[t,T]$. *Then, for the iterated Stratonovich stochastic integral*

$$I_{T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad(k=5,6)$$

the following expansion

$$I_{T,t}^{*(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\quad(k=5,6)$$

is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_k\ldots j_1}=\int\limits_t^{T}\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k\quad(k=5,6);$$

another notations as in Theorem 1.

Theorem 12 **[Kuz1]** (2025). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a CONS of Legendre polynomials or trigonometric Fourier basis in* $L_2[t,T]$*. Then, for the iterated Stratonovich stochastic integral*

$$I_{T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\quad (k=7,8)$$

the following expansion

$$I_{T,t}^{*(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\quad (k=7,8)$$

is valid, where $i_1,\ldots,i_8=0,1,\ldots,m,$

$$C_{j_k\ldots j_1}=\int\limits_t^{T}\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k\quad (k=7,8);$$

another notations as in Theorem 1.

Sketch of the proof of Theorem 10

The condition (27) will be satisfied under the conditions of Theorem 12 if

$$\left|\sum_{j=0}^{p}\int_{\tau}^{s}(x-t)^{l}\phi_j(x)dx\cdot\int_{u}^{\theta}(x-t)^{m}\phi_j(x)dx\right|\le K, \tag{32}$$

$$\left|\sum_{j_1=0}^{p}\int_{\tau}^{s}(x-t)^{l}\phi_j(x)\int_{\tau}^{x}(y-t)^{m}\phi_j(y)dydx\right|\le K, \tag{33}$$

where $p\in\mathbf{N}$, $l,m=0,1,2,\ldots,$ $t\le\tau<s\le T$, $t\le u<\theta\le T$, constant K does not depend on p,s,τ,u,θ (but only on t, T).

The conditions (32),(33) are proved using the Cauchy–Bunyakovsky inequality, Fubini's Theorem and Parseval's equality.

Sketch of the proof of Theorem 11

Denote

$$C_{j_k\ldots j_1}(s,\tau)=\int\limits_{\tau}^{s}\phi_{j_k}(t_k)\ldots\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k\quad (k=\overline{1,6},\ t\le\tau<s\le T)$$

The condition (27) will be satisfied if

$$\left|\sum_{j_1=0}^{p}C_{j_1j_1}(s,\tau)\right|\le K,\quad \left|\sum_{j_1=0}^{p}C_{j_1}(s,\tau)C_{j_1}(\theta,u)\right|\le K, \qquad (34)$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_1j_1}(s,\tau)C_{j_2}(\theta,u)\right|\le K,\quad \left|\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_1}(s,\tau)C_{j_2}(\theta,u)\right|\le K, \quad (35)$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1}(s,\tau)C_{j_1}(\theta,u)\right|\le K,\quad \left|\sum_{j_1,j_2=0}^{p}C_{j_1j_1}(s,\tau)C_{j_2j_2}(\theta,u)\right|\le K, \quad (36)$$

$$\left|\sum_{j_1,j_2=0}^{p}C_{j_2j_1}(s,\tau)C_{j_2j_1}(\theta,u)\right|\le K,\quad \left|\sum_{j_1,j_2=0}^{p}C_{j_1}(s,\tau)C_{j_1}(\rho,v)C_{j_2j_2}(\theta,u)\right|\le K, \quad (37)$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s,\tau)C_{j_1j_2}(\theta,u)\right|\le K,\quad \left|\sum_{j_1,j_2=0}^{p} C_{j_1}(s,\tau)C_{j_2}(\rho,v)C_{j_1j_2}(\theta,u)\right|\le K, \quad (38)$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_2j_1j_1}(s,\tau)\right|\le K, \quad (39)$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s,\tau)\right|\le K,\quad \left|\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_2j_1}(s,\tau)\right|\le K, \quad (40)$$

where $p\in\mathbf{N}$, $t\le\tau<s\le T$, $t\le u<\theta\le T$, $t\le v<\rho\le T$, constant K does not depend on p,s,τ,u,θ,v,ρ (but only on t,T).

The conditions (34)–(38) are proved using the Cauchy–Bunyakovsky inequality, Fubini's Theorem and Parseval's equality for one and two functions. To prove (39),(40) we use the equality (that follows from Fubini's Theorem for the case $\psi_1(\tau),\ldots,\psi_4(\tau)\equiv 1$)

$$C_{j_4j_3j_2j_1}(s,\tau)+C_{j_1j_2j_3j_4}(s,\tau)=$$

$$=C_{j_4}(s,\tau)C_{j_3j_2j_1}(s,\tau)-C_{j_3j_4}(s,\tau)C_{j_2j_1}(s,\tau)+C_{j_2j_3j_4}(s,\tau)C_{j_1}(s,\tau). \quad (41)$$

Substitute $j_4 = j_3$, $j_2 = j_1$ into (41)

$$C_{j_3j_3j_1j_1}(s,\tau) + C_{j_1j_1j_3j_3}(s,\tau) =$$

$$= C_{j_3}(s,\tau)C_{j_3j_1j_1}(s,\tau) - C_{j_3j_3}(s,\tau)C_{j_1j_1}(s,\tau) + C_{j_1j_3j_3}(s,\tau)C_{j_1}(s,\tau). \quad (42)$$

Since

$$\boxed{\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau) = \sum_{j_1,j_3=0}^{p} C_{j_1j_1j_3j_3}(s,\tau)},$$

then

$$2\sum_{j_1,j_3=0}^{p} C_{j_3j_3j_1j_1}(s,\tau) = 2\sum_{j_1,j_3=0}^{p} C_{j_3}(s,\tau)C_{j_3j_1j_1}(s,\tau) - \left(\sum_{j_1=0}^{p} C_{j_1j_1}(s,\tau)\right)^2. \quad (43)$$

Using (43), Cauchy–Bunyakovsky's inequality and Parseval's equality, we obtain (39). Analogously we obtain (40), since

$$\boxed{\sum_{j_1,j_3=0}^{p} C_{j_3j_1j_3j_1}(s,\tau) = \sum_{j_1,j_3=0}^{p} C_{j_1j_3j_1j_3}(s,\tau), \sum_{j_1,j_3=0}^{p} C_{j_1j_3j_3j_1}(s,\tau) = \sum_{j_1,j_3=0}^{p} C_{j_3j_1j_1j_3}(s,\tau)}$$

Sketch of the proof of Theorem 12

The condition (27) will be satisfied if (34)–(40) together with

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_1}(s,\tau)C_{j_2}(\rho,v)C_{j_1}(\theta,u)C_{j_2}(\mu,w)\right| \le K,$$

$$\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p} C_{j_{d_1}j_{d_1-1}j_{d_1-2}j_{d_1-3}j_{d_1-4}j_{d_1-5}}(s,\tau)\Big|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right| \le K, \quad (44)$$

$$\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p} \left(C_{j_{d_2}j_{d_2-1}j_{d_2-2}j_{d_2-3}j_{d_2-4}}(s,\tau)C_{j_{d_1}}(\theta,u)\right)\Big|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right| \le K,$$

$$\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p} \left(C_{j_{d_2}j_{d_2-1}j_{d_2-2}j_{d_2-3}}(s,\tau)C_{j_{d_1}j_{d_1-1}}(\theta,u)\right)\Big|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right| \le K,$$

$$\left|\sum_{j_{g_1},j_{g_3},j_{g_5}=0}^{p} \left(C_{j_{d_2}j_{d_2-1}j_{d_2-2}}(s,\tau)C_{j_{d_1}j_{d_1-1}j_{d_1-2}}(\theta,u)\right)\Big|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right| \le K,$$

$$\left| \sum_{j_{g_1}, j_{g_3}, j_{g_5}=0}^{p} \left(C_{j_{d_3} j_{d_3-1} j_{d_3-2} j_{d_3-3}}(s,\tau) C_{j_{d_2}}(\theta,u) C_{j_{d_1}}(\rho,v) \right) \Bigg|_{j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}} \right| \le$$

$$\le K,$$

$$\left| \sum_{j_{g_1}, j_{g_3}, j_{g_5}=0}^{p} \left(C_{j_{d_3} j_{d_3-1} j_{d_3-2}}(s,\tau) C_{j_{d_2} j_{d_2-1}}(\theta,u) C_{j_{d_1}}(\rho,v) \right) \Bigg|_{j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}} \right| \le$$

$$\le K,$$

$$\left| \sum_{j_{g_1}, j_{g_3}, j_{g_5}=0}^{p} \left(C_{j_{d_3} j_{d_3-1}}(s,\tau) C_{j_{d_2} j_{d_2-1}}(\theta,u) C_{j_{d_1} j_{d_1-1}}(\rho,v) \right) \Bigg|_{j_{g_1}=j_{g_2}, j_{g_3}=j_{g_4}, j_{g_5}=j_{g_6}} \right| \le$$

$$\le K$$

are fulfilled, where $p \in \mathbf{N}$, $\tau < s$, $u < \theta$, $v < \rho$, $w < \mu$, constant K does not depend on $p, s, \tau, u, \theta, v, \rho, w, \mu \in [t, T]$ (but only on t, T).

All of the above inequalities (except (44)) are proved using Cauchy–Bunyakovsky's inequality, Parseval's equality and Fubini's Theorem. The main difficulty is related to the proof of (44).

It is easy to see that (44) reduces to the following 15 inequalities

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1}(s,\tau)\right| \le K, \qquad \left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_2j_3j_2j_1}(s,\tau)\right| \le K,$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_3j_1j_2j_1}(s,\tau)\right| \le K, \qquad \left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_3j_2j_1}(s,\tau)\right| \le K,$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_2j_3j_3j_1}(s,\tau)\right| \le K, \qquad \left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_2j_2j_1j_1}(s,\tau)\right| \le K,$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_2j_1j_1}(s,\tau)\right| \le K, \qquad \left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_3j_2j_1j_1}(s,\tau)\right| \le K,$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_2j_1j_2j_1}(s,\tau)\right| \le K, \qquad \left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2j_2j_1}(s,\tau)\right| \le K,$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1j_3j_3j_2j_1}(s,\tau)\right| \le K, \qquad \left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2j_3j_2j_1}(s,\tau)\right| \le K,$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_1j_3j_2j_1}(s,\tau)\right| \le K, \qquad \left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_3j_2j_2j_1}(s,\tau)\right| \le K,$$

$$\left|\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2j_1}(s,\tau)\right| \le K,$$

where $p \in \mathbf{N}$, $t \le \tau < s \le T$, constant K does not depend on p, s, τ (but only on t, T).

All of the above inequalities (except the red ones) are proved using Cauchy–Bunyakovsky's inequality, Parseval's equality, Fubini's Theorem as well as the following equality

$$C_{j_6j_5j_4j_3j_2j_1}(s,\tau)+C_{j_1j_2j_3j_4j_5j_6}(s,\tau)=$$
$$=C_{j_6}(s,\tau)C_{j_5j_4j_3j_2j_1}(s,\tau)-C_{j_5j_6}(s,\tau)C_{j_4j_3j_2j_1}(s,\tau)+$$
$$+C_{j_4j_5j_6}(s,\tau)C_{j_3j_2j_1}(s,\tau)-C_{j_3j_4j_5j_6}(s,\tau)C_{j_2j_1}(s,\tau)+$$
$$+C_{j_2j_3j_4j_5j_6}(s,\tau)C_{j_1}(s,\tau) \qquad (45)$$

that follows from Fubini's Theorem for the case $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$. At that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$. For example,

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_3j_1j_2j_1}(s,\tau)=\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_3}(s,\tau)C_{j_2j_3j_1j_2j_1}(s,\tau)-$$
$$-C_{j_2j_3}(s,\tau)C_{j_3j_1j_2j_1}(s,\tau)+C_{j_3j_2j_3}(s,\tau)C_{j_1j_2j_1}(s,\tau)-$$
$$-C_{j_1j_3j_2j_3}(s,\tau)C_{j_2j_1}(s,\tau)+C_{j_2j_1j_3j_2j_3}(s,\tau)C_{j_1}(s,\tau)\Biggr).$$

The remaining inequalities (the red ones) are proved using the Cauchy–Bunyakovsky inequality, Fubini's Theorem, Parseval's equality, Lebesgue's Dominated Convergence Theorem and estimates of integrals of $\phi_j(x)$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a CONS of Legendre polynomials or trigonometric Fourier basis in $L_2[t, T]$. For example,

$$\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_3 j_1 j_3 j_2 j_1}(s,\tau)\right)^2 = \left(\sum_{j_2=0}^{p} 1 \cdot \sum_{j_1,j_3=0}^{p} C_{j_2 j_3 j_1 j_3 j_2 j_1}(s,\tau)\right)^2 \le$$

$$\le \sum_{j_2=0}^{p} 1^2 \cdot \sum_{j_2=0}^{p} \left(\sum_{j_1,j_3=0}^{p} C_{j_2 j_3 j_1 j_3 j_2 j_1}(s,\tau)\right)^2 =$$

$$= (p+1) \sum_{j_2=0}^{p} \left(\sum_{j_1,j_3=0}^{p} C_{j_2 j_3 j_1 j_3 j_2 j_1}(s,\tau)\right)^2 =$$

$$= (p+1) \sum_{j_2=0}^{p} \left(\sum_{j_1,j_3=0}^{p} \int_{\tau}^{s} \phi_{j_2}(t_6) \int_{\tau}^{t_6} \phi_{j_2}(t_2) C_{j_1}(t_2,\tau) C_{j_3 j_1 j_3}(t_6,t_2) dt_2 dt_6\right)^2 \le$$

$$\leq (p+1) \sum_{j_2, j_2'=0}^{p} \left(\sum_{j_1, j_3=0}^{p} \int_{\tau}^{s} \phi_{j_2}(t_6) \int_{\tau}^{t_6} \phi_{j_2'}(t_2) C_{j_1}(t_2, \tau) C_{j_3 j_1 j_3}(t_6, t_2) dt_2 dt_6 \right)^2 \leq$$

$$\leq (p+1) \sum_{j_2, j_2'=0}^{\infty} \left(\int_{\tau}^{s} \phi_{j_2}(t_6) \int_{\tau}^{t_6} \phi_{j_2'}(t_2) \sum_{j_1, j_3=0}^{p} C_{j_1}(t_2, \tau) C_{j_3 j_1 j_3}(t_6, t_2) dt_2 dt_6 \right)^2 =$$

$$= (p+1) \int_{\tau}^{s} \int_{\tau}^{t_6} \left(\sum_{j_1=0}^{p} C_{j_1}(t_2, \tau) \sum_{j_3=0}^{p} C_{j_3 j_1 j_3}(t_6, t_2) \right)^2 dt_2 dt_6 =$$

$$= (p+1) \int_{\tau}^{s} \int_{\tau}^{t_6} \left(\sum_{j_1=0}^{p} C_{j_1}(t_2, \tau) \sum_{j_3=p+1}^{\infty} C_{j_3 j_1 j_3}(t_6, t_2) \right)^2 dt_2 dt_6 \leq$$

$$\leq (p+1) \int_{\tau}^{s} \int_{\tau}^{t_6} \sum_{j_1=0}^{p} C_{j_1}^2(t_2, \tau) \sum_{j_1=0}^{p} \left(\sum_{j_3=p+1}^{\infty} C_{j_3 j_1 j_3}(t_6, t_2) \right)^2 dt_2 dt_6 \leq$$

$$\leq (p+1)\int_{\tau}^{s}\int_{\tau}^{t_6}\sum_{j_1=0}^{\infty}C_{j_1}^2(t_2,\tau)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3}(t_6,t_2)\right)^2 dt_2dt_6 =$$

$$\leq (p+1)\int_{\tau}^{s}\int_{\tau}^{t_6}(t_2-\tau)\sum_{j_1=0}^{p}\left(\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3}(t_6,t_2)\right)^2 dt_2dt_6 =$$

$$=(p+1)\int_{\tau}^{s}\int_{\tau}^{t_6}(t_2-\tau)\sum_{j_1=0}^{p}\left(\int_{t_2}^{t_6}\phi_{j_1}(\theta)\sum_{j_3=p+1}^{\infty}C_{j_3}(\theta,t_2)C_{j_3}(t_6,\theta)d\theta\right)^2 dt_2dt_6 \leq$$

$$\leq (p+1)\int_{\tau}^{s}\int_{\tau}^{t_6}(t_2-\tau)\sum_{j_1=0}^{\infty}\left(\int_{t_2}^{t_6}\phi_{j_1}(\theta)\sum_{j_3=p+1}^{\infty}C_{j_3}(\theta,t_2)C_{j_3}(t_6,\theta)d\theta\right)^2 dt_2dt_6 =$$

$$=(p+1)\int_{\tau}^{s}\int_{\tau}^{t_6}(t_2-\tau)\int_{t_2}^{t_6}\left(\sum_{j_3=p+1}^{\infty}C_{j_3}(\theta,t_2)C_{j_3}(t_6,\theta)\right)^2 d\theta dt_2dt_6. \quad (46)$$

For the trigonometric case, we have

$$\left|C_j(x,v)\right| = \left|\int\limits_v^x \phi_j(\tau)d\tau\right| < \frac{C}{j} \quad (j>0), \tag{47}$$

where constant C does not depend on j, x, v. Moreover,

$$\sum_{j=p+1}^{\infty} \frac{1}{j^2} \le \int\limits_p^{\infty} \frac{dx}{x^2} = \frac{1}{p}. \tag{48}$$

Combining (46)–(48), we get

$$\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_3 j_1 j_3 j_2 j_1}(s,\tau)\right)^2 \le \frac{K_1(p+1)}{p^2} \le K^2,$$

where constants K, K_1 depend only on t, T.

Appendix 1

Consider an example on proof of Step 3 in the proof of Theorem 9. Let us prove that

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p} C_{j_3 j_4 j_4 j_3 j_1 j_1} =$$

$$= \lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\int\limits_t^T \psi_6(t_6)\phi_{j_3}(t_6)\int\limits_t^{t_6}\psi_5(t_5)\phi_{j_4}(t_5)\int\limits_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\times$$

$$\times\int\limits_t^{t_4}\psi_3(t_3)\phi_{j_3}(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 = 0, \tag{49}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2[t,T]$ and $\psi_1(\tau),\ldots,\psi_6(\tau)\in L_2[t,T]$.

Step A. Using Step 2 ($k=2$) and generalized Parseval's equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_3,j_4=0}^{p}\int_t^T\psi_6(t_6)\phi_{j_3}(t_6)\int_t^{\boxed{T}}\psi_5(t_5)\phi_{j_4}(t_5)\int_t^{t_5}\psi_4(t_4)\phi_{j_4}(t_4)\int_t^{\boxed{T}}\psi_3(t_3)\phi_{j_3}(t_3)\times$$

$$\times\int_t^{\boxed{T}}\psi_2(t_2)\phi_{j_1}(t_2)\int_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6= \qquad (50)$$

$$=\int_t^T\psi_6(t_6)\psi_3(t_6)dt_6\cdot\frac{1}{2}\int_t^T\psi_5(t_4)\psi_4(t_4)dt_4\cdot\frac{1}{2}\int_t^T\psi_2(t_2)\psi_1(t_2)dt_2. \qquad (51)$$

Rewrite (51) in the form

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_4<t_5\}}\psi_6(t_6)\phi_{j_3}(t_6)\psi_5(t_5)\phi_{j_4}(t_5)\psi_4(t_4)\phi_{j_4}(t_4)\times$$

$$\times\psi_3(t_3)\phi_{j_3}(t_3)\psi_2(t_2)\phi_{j_1}(t_2)\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6. \qquad (52)$$

Step B. From (52) we obtain

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_4<t_5\}}s_q(t_2,t_3,t_4)\psi_6(t_6)\psi_5(t_5)\psi_1(t_1)\phi_{j_3}(t_6)\phi_{j_3}(t_3)\times$$

$$\times\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}s_q(t_2,t_6,t_4)\psi_6(t_6)\psi_5(t_4)\psi_1(t_2)dt_2dt_4dt_6, \qquad (53)$$

where $s_q(t_2,t_3,t_4)$ is a partial sum of the Fourier series for the function $g(t_2,t_3,t_4)=\psi_2(t_2)\psi_3(t_3)\psi_4(t_4)\mathbf{1}_{\{t_2<t_3\}}$.

• **Remark.** Note that the equality (53) remains true when s_q is a partial sum of the Fourier series of any function from $L_2([t,T]^3)$, i.e. the equality holds on a dense subset in $L_2([t,T]^3)$.

• **Remark.** The right-hand side of (53) defines (as a scalar product of s_q and $\frac{1}{4}\psi_6\psi_5\psi_1$ in $L_2([t,T]^3)$) a linear continuous functional in $L_2([t,T]^3)$ given by $\frac{1}{4}\psi_6\psi_5\psi_1$. On the left-hand side of (53) (by virtue of (53)) there is a linear continuous functional on a dense subset in $L_2([t,T]^3)$. This functional can be uniquely extended to a linear continuous functional in $L_2([t,T]^3)$.

Let us implement the passage to the limit $\lim\limits_{q\to\infty}$ in (53)

$$\sum_{j_1,j_3,j_4=0}^{\infty}\int\limits_{[t,T]^6}\mathbf{1}_{\{t_1<\underline{t_2<t_3}\}}\mathbf{1}_{\{t_4<t_5\}}\psi_6(t_6)\psi_5(t_5)\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\times$$

$$\times\phi_{j_3}(t_6)\phi_{j_3}(t_3)\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{4}\int\limits_{[t,T]^3}\mathbf{1}_{\{t_2<t_6\}}\psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6.$$

After two more steps (by analogy with **Step B**) we obtain

$$\sum_{j_1,j_3,j_4=0}^{\infty} \int\limits_{[t,T]^6} \mathbf{1}_{\{t_1<\underline{t_2<t_3<t_4}<\underline{t_5<t_6}\}} \psi_6(t_6)\psi_5(t_5)\psi_4(t_4)\psi_3(t_3)\psi_2(t_2)\psi_1(t_1)\times$$

$$\times\phi_{j_3}(t_6)\phi_{j_3}(t_3)\phi_{j_4}(t_5)\phi_{j_4}(t_4)\phi_{j_1}(t_2)\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{4} \int\limits_{[t,T]^3} \mathbf{1}_{\{t_2<t_6\}} \underbrace{\mathbf{1}_{\{t_6<t_4\}}\mathbf{1}_{\{t_4<t_6\}}}_{=0} \psi_6(t_6)\psi_3(t_6)\psi_5(t_4)\psi_4(t_4)\times$$

$$\times\psi_2(t_2)\psi_1(t_2)dt_2dt_4dt_6 = 0.$$

The equality (49) is proved.

Appendix 2

Consider step 4 in the proof of Theorem 9. Using Fubini's Theorem, we get

$$\int_t^T h_k(t_k)\ldots\int_t^{t_{l+2}} h_{l+1}(t_{l+1})\int_t^{t_{l+1}} h_l(t_l)\int_t^{t_l} h_{l-1}(t_{l-1})\ldots\int_t^{t_2} h_1(t_1)dt_1\ldots$$

$$\ldots dt_{l-1}dt_l dt_{l+1}\ldots dt_k =$$

$$= \int_t^T h_k(t_k)\ldots\int_t^{t_{l+2}} h_{l+1}(t_{l+1})\left(\int_t^{t_{l+1}} h_l(t_l)dt_l\right)\int_t^{t_{l+1}} h_{l-1}(t_{l-1})\ldots$$

$$\ldots\int_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-1}dt_{l+1}\ldots dt_k -$$

$$-\int_t^T h_k(t_k)\ldots\int_t^{t_{l+2}} h_{l+1}(t_{l+1})\int_t^{t_{l+1}} h_{l-1}(t_{l-1})\left(\int_t^{t_{l-1}} h_l(t_l)dt_l\right)\int_t^{t_{l-1}} h_{l-2}(t_{l-2})\ldots$$

$$\ldots\int_t^{t_2} h_1(t_1)dt_1\ldots dt_{l-2}dt_{l-1}dt_{l+1}\ldots dt_k, \qquad (54)$$

where $2 < l < k-1$ and $h_1(\tau), \ldots, h_k(\tau) \in L_2[t,T]$. The case $l = k$ is considered by analogy with (54). The case $l = 1$ is obvious.

Suppose that $k > 2r$. Let us carry out the transformation (54) for

$$C_{j_k \ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$$

iteratively for $j_{q_1}, \ldots, j_{q_{k-2r}}$ $(k > 2r)$. As a result, we obtain

$$C_{j_k \ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \sum_{d=1}^{2^{k-2r}} (-1)^{d-1} \left(\hat{C}^{(d)}_{j_k \ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} - \bar{C}^{(d)}_{j_k \ldots j_1}\Big|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \right), \tag{55}$$

where some terms in the sum

$$\sum_{d=1}^{2^{k-2r}}$$

can be identically equal to zero.

Applying (55) and **Step 3** ($k=2r$) in the proof of Theorem 9, we get

$$\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$=\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\times$$

$$\times\lim_{p\to\infty}\sum_{j_{g_1},j_{g_3},\ldots j_{g_{2r-1}}=0}^{p}\left(\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}-\bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\right)=$$

$$=\sum_{d=1}^{2^{k-2r}}(-1)^{d-1}\frac{1}{2^r}\prod_{l=1}^{r}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\times$$

$$\times\left(\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}-\bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}\right). \tag{56}$$

Case A. The condition

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 1 \tag{57}$$

is fulfilled for

$$\hat{C}_{j_k \ldots j_1}^{(d)}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},\ \bar{C}_{j_k \ldots j_1}^{(d)}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \ (d=1,2,\ldots,2^{k-2r}), \tag{58}$$

$$C_{j_k \ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{59}$$

Case B. The condition (57) is satisfied for (58) and the condition

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 0 \tag{60}$$

is fulfilled for (59).

Case C. The condition (60) is satisfied for (58), (59).

For **Case A**, using transformation (54), we obtain

$$\sum_{d=1}^{2^{k-2r}} \frac{(-1)^{d-1}}{2^r}\left(\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)} - \bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}\right)$$

$$= \frac{1}{2^r}\, C_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}. \qquad (61)$$

For **Case B**

$$\hat{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)} = \bar{C}^{(d)}_{j_k\ldots j_1}\Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot)}. \qquad (62)$$

For **Case C**

$$\prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} = 0. \qquad (63)$$

Combining (61), (62), (63) and (56), we complete the proof of **Step 4** $(k > 2r)$ in the proof of Theorem 9.

Thanks for your attention!