*Journal of Mathematical Sciences, Vol. 99, No. 2, 2000*

# THE UNIFIED TAYLOR–ITO EXPANSION

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UDC 519.2

*We consider the problem of the Taylor–Ito expansion for Ito processes in a neighborhood of a fixed time moment. The Taylor–Ito expansion known in literature is unified by a canonical system of repeated stochastic Ito integrals with polynomial weight functions. The unified expansion has some computational advantages, such as recurrent relations between the expansion coefficients, ordering of the expansion with respect to smallness of its terms, and a smaller number of applied repeated stochastic integrals of different types. The unified expansion is more convenient in constructing algorithms of numerical solution for stochastic Ito differential equations. Bibliography: 11 titles.*

## 1. Introduction

This paper is devoted to the problem of the Taylor–Ito series expansion for Ito processes. This problem is relatively recent in the theory of random processes; the first publications concerning the problem appeared in the 1970s and 1980s (Milstein, 1974; Wagner and Platen, 1978 and 1982; Platen, 1981 and 1982). In [3, 4], Wagner and Platen were the first to introduce and apply a Taylor–Ito expansion, i.e., an expansion of a smooth inertialess nonlinear transformation of a solution of a stochastic Ito differential equation into a series in repeated stochastic integrals with application of the Ito formula.

In this paper, we apply the authors' results [7, 9, 10] in construction of the unified Taylor–Ito expansion. Let us explain what we have in mind. It is possible to reduce repeated stochastic integrals from the Taylor–Ito expansion in [3, 4] to a system of canonical repeated stochastic Ito integrals of smaller multiplicity with polynomial integrands. In [7, 10], these transformations are based on formulas of the change of the integration order for repeated stochastic Ito integrals obtained by the authors [9]. The result of collecting similar terms is called the unified Taylor–Ito expansion. It is important to note that the coefficients of the unified Taylor–Ito expansion are determined by recurrent relations. Another important property of the unified Taylor–Ito expansion is that it contains a significantly lesser number of different repeated stochastic integrals than the Taylor–Ito expansion in the form of Wagner and Platen [3, 4]. To make a comparison, let us note that the unified Taylor–Ito expansion up to terms of third order of smallness contains 12 different repeated stochastic integrals, while the similar Taylor–Ito expansion in the form of Wagner and Platen contains 17 different repeated stochastic integrals.

In addition, the more terms of expansion are taken into account, the more apparent is the mentioned difference. This advantage of the unified Taylor–Ito expansion is especially important since approximation of repeated stochastic integrals is a complicated theoretical and computational problem.

## 2. Definitions and assumptions

Consider a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbf{f}_t = \mathbf{f}(t,\omega) \in R_m$ be a Wiener process such that

$$M\{d\mathbf{f}_t d\mathbf{f}_t^T\} = \Sigma_f^2 dt,$$

$$\Sigma_f^2 = \text{diag}\{\sigma_{f_1}^2,\ \sigma_{f_2}^2, \ldots, \sigma_{f_m}^2\};\ \ \sigma_{f_i}^2 < \infty,\ \ i = 1, 2, \ldots, m,$$

where $M\{\cdot\}$ is the operator of mathematical expectation.

Consider a system of stochastic Ito differential equations of the following form:

$$d\mathbf{x}_t = \mathbf{a}(\mathbf{x}_t, t)dt + \Sigma(\mathbf{x}_t, t)d\mathbf{f}_t,\ \ \mathbf{x}_0 = \mathbf{x}(0), \tag{1}$$

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 244, 1997, pp. 186–204. Original article submitted December 15, 1997.



1072-3374/00/9902-1130 $25.00 

where $\mathbf{x}_t = \mathbf{x}(t,\omega) \in R_n$ is a solution of Eq. (1). It is assumed that the functions $\mathbf{a}(\mathbf{x},t) \in R_n$ and $\Sigma(\mathbf{x},t) \in R_{n\times m}$ are multiply continuously differentiable with respect to both arguments and satisfy the conditions of existence and uniqueness of a solution for Eq. (1).

We say that a random process $g_t \in R_1$ is adapted with respect to the Wiener process $\mathbf{f}_t \in R_m$ on $[a,b]$ if, for any times $\tau, t, s \in [a,b]$ such that $\tau \le t < s$, the values $g_\tau$ are stochastically independent with the differences $\mathbf{f}_s^{(i)} - \mathbf{f}_t^{(i)} (i = 1,\dots,m)$, where $\mathbf{f}_t^{(i)}$ is the $i$th component of the Wiener process $\mathbf{f}_t$.

We say that a process $g_t$ is continuous in mean of degree $m$ on $[a,b]$ if the condition

$$\lim_{t\to\tau} M\{\mid g_t - g_\tau \mid^m\} = 0$$

is fulfilled for any times $t, \tau \in [a,b]$.

Consider a partition $a = t_0, t_1, t_2, \dots, t_N = b$ of a segment $[a,b]$ such that $t_i < t_{i+1}$ for $0 \le i \le N-1$. Denote $\Delta_N = \max_{0\le i\le N-1} |t_{i+1} - t_i|$.

The mean-square limit

$$\int_a^b g_t d\mathbf{f}_t^{(i)} q_t = \underset{\Delta_N\to 0}{\text{l.i.m.}} \sum_{k=0}^{N-1} g_{t_k} \left(\mathbf{f}_{t_{k+1}}^{(i)} - \mathbf{f}_{t_k}^{(i)}\right) q_{t_{k+1}} \tag{2}$$

is called the generalized stochastic Ito integral of random processes $g_t$ and $q_t$ with respect to the Wiener process $\mathbf{f}_t^{(i)} \in R_1 (i = 1,\dots,m)$ (see Stratonovich [1]).

Assume that the processes $q_t$ and $g_t$ satisfy the following conditions.

(A1) $q_t \equiv 1$.

(A2) The process $g_t$ is adapted with respect to the Wiener process $\mathbf{f}_t^{(i)} \in R_1 (i = 1,\dots,m)$.

(A3) The process $g_t$ is mean-square continuous on $[a,b]$.

(A4) $M\{g_t^2\} < \infty$ for all $t \in [a,b]$.

It is easy to show that the conditions just formulated imply the existence of the generalized stochastic integral (2). It is also easy to show that the same conditions imply the existence of a stochastic integral of the form

$$\int_a^b g_t dt q_t = \underset{\Delta_N\to 0}{\text{l.i.m.}} \sum_{k=0}^{N-1} g_{t_k} (t_{k+1} - t_k)\, q_{t_{k+1}}. \tag{3}$$

**Definition 1.** *We say that a process $\eta_s = R(\mathbf{x}_s, s)$ is Ito continuously differentiable in the mean-square sense for $s \in [0,T]$ along trajectories of Eq. (1) if the representation*

$$\eta_s = \eta_t + \int_t^s B_0\{R(\mathbf{x}_\tau,\tau)\}d\tau + \sum_{i=1}^m \int_t^s B_1^{(i)}\{R(\mathbf{x}_\tau,\tau)\}d\mathbf{f}_\tau^{(i)} \tag{4}$$

*takes place for all $s, t \in [0,T]$ such that $s \ge t$ with probability 1, and the integrals on the right in (4) exist in the mean-square sense.*

*In formula (4), $\mathbf{x}_\tau$ is a solution of Eq. (1) and $B_0\{R(\mathbf{x}_\tau,\tau)\}$ and $B_1^{(i)}\{R(\mathbf{x}_\tau,\tau)\}$, $i = 1,\dots,m$, are processes continuous in the mean-square sense on $[0,T]$ called the systematic and diffusion Ito derivatives of the process $\eta_s$, respectively.*

**Lemma 1** (Ito formula). *Assume that*

*(1°) the partial derivatives $\frac{\partial}{\partial t}R(\mathbf{x},t)$ and $\frac{\partial}{\partial \mathbf{x}^{(i)}}R(\mathbf{x},t)\frac{\partial^2}{\partial \mathbf{x}^{(i)}\partial \mathbf{x}^{(j)}}R(\mathbf{x},t)$, $i,j = 1,2,\dots,n$, exist and are continuous on $R_n \times [0,T]$;*

*(2°) the functions $\mathbf{a}^{(i)}(\mathbf{x},t)$ and $\Sigma^{(ij)}(\mathbf{x},t)$ and the processes $\mathbf{a}^{(i)}(\mathbf{x}_t,t)$ and $\Sigma^{(ij)}(\mathbf{x}_t,t)$ $(i = 1,\dots,n,\ j = 1,\dots,m)$ have the following property: the processes $L\{R(\mathbf{x}_t,t)\}$ and $G_0^{(j)}\{R(\mathbf{x}_t,t)\}$, $j = 1,\dots,m$, satisfy conditions (A3) and (A4).*

*Then the process $\eta_s = R(\mathbf{x}_s,s)$ is Ito continuously differentiable in the mean-square sense on $[0,T]$, and its derivatives $B_0\{R(\mathbf{x}_t,t)\}$ and $B_1^{(i)}\{R(\mathbf{x}_t,t)\}$, $i = 1,\dots,m$, are given by the following formulas:*

$$B_0\{R(\mathbf{x}_t,t)\} = L\{R(\mathbf{x}_t,t)\}, B_1^{(i)}\{R(\mathbf{x},t)\} = G_0^{(i)}\{R(\mathbf{x},t)\}, \tag{5}$$

*where*

$$L\{\cdot\} = \frac{\partial\{\cdot\}}{\partial t} + \sum_{i=1}^{n} a^{(i)}(\mathbf{x},t)\frac{\partial\{\cdot\}}{\partial \mathbf{x}^{(i)}} + \frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n} \sigma_{f_j}^2 \Sigma^{(lj)}(\mathbf{x},t)\Sigma^{(ij)}(\mathbf{x},t)\frac{\partial^2\{\cdot\}}{\partial \mathbf{x}^{(l)}\partial \mathbf{x}^{(i)}} \tag{6}$$

*and*

$$G_0^{(i)}\{\cdot\} = \sum_{j=1}^{n} \Sigma^{(ji)}(\mathbf{x},t)\frac{\partial\{\cdot\}}{\partial \mathbf{x}^{(j)}}, i = 1, \dots, m. \tag{7}$$

*In addition, the equality*

$$\eta_s = \eta_t + \int_t^s L\{R(\mathbf{x}_\tau, \tau)\}\, d\tau + \sum_{i=1}^{m} \int_t^s G_0^{(i)}\{R(\mathbf{x}_\tau, \tau)\}\, d\mathbf{f}_\tau^{(i)}$$

*holds for all $s, t \in [0,T]$ such that $s \geq t$ with probability 1, and the integrals on the right exist in the mean-square sense.*

We call a family of $k$-index elements a $k$-rank matrix ${}^{(k)}A$, i.e., ${}^{(k)}A = \|A^{(i_1\dots i_k)}\|_{i_1,\dots,i_k=1}^{m_1,\dots,m_k}$. Thus, ${}^{(0)}A$ is a scalar value, ${}^{(1)}A$ is an $m_1 \times 1$-matrix, ${}^{(2)}A$ is an $m_1 \times m_2$-matrix, and so on. The symbol ${}^{(k)}A = \|{}^{(k-1)}A^{(i_1)}\|_{i_1=1}^{m_1}$, where ${}^{(k-1)}A^{(i_1)} = \|A^{(i_1\dots i_k)}\|_{i_2,\dots,i_k=1}^{m_2,\dots,m_k}$, denotes a block matrix such that its elements are of rank $k-1$. Below we sometimes omit the rank of scalars and column matrices.

The matrix ${}^{(k)}C$ defined by the formula

$${}^{(k)}C = \left\| \sum_{i_1,\dots,i_l=1}^{m_1,\dots,m_l} A^{(i_1\dots i_{k+l})}B^{(i_1\dots i_l)} \right\|_{i_{l+1},\dots,i_{l+k}=1}^{m_{l+1},\dots,m_{l+k}}$$

is called the convolution of matrices ${}^{(k+l)}\mathrm{A}$ and ${}^{(l)}B$; below we denote this matrix by ${}^{(k)}C = {}^{(k+l)}A^{(l)}{}^{(l)}B$.

**Definition 2.** *A process $\eta_s = R(\mathbf{x}_s, s)$ is called $N$-times Ito continuously differentiable in the mean-square sense on $[0,T]$ along trajectories of Eq. (1) if the representation*

$${}^{(r_l)}B_{\gamma_l\dots\gamma_1}\{R(\mathbf{x}_s, s)\} = {}^{(r_l)}B_{\gamma_l\dots\gamma_1}\{R(\mathbf{x}_t, t)\} + \int_t^s {}^{(r_l)}B_{0\gamma_l\dots\gamma_1}\{R(\mathbf{x}_\tau, \tau)\}d\tau$$
$$+ \int_t^s {}^{(r_l+1)}B_{1\gamma_l\dots\gamma_1}\{R(\mathbf{x}_\tau, \tau)\} \overset{1}{\cdot}\, d\mathbf{f}_\tau$$

*is fulfilled for all $l = 0, 1, \dots, N-1$, and $s, t \in [0,T]$ such that $s \geq t$ with probability 1 and has the following additional properties: the integrals on the right exist in the mean-square sense; $\gamma_q = 0, 1$; $r_l = \sum_{i=1}^{l} \gamma_i$; ${}^{(r_q)}B_{\gamma_q\dots\gamma_1}\{R(\mathbf{x}_t, t)\}$, $q = 1, 2, \dots N$, are matrix random processes continuous in the mean-square sense. These processes are called the Ito $q$th derivatives of the process $\eta_s$ in the mean-square sense along trajectories of system (1); ${}^{(0)}B_{\underbrace{0\dots0}_{q}}\{R(\mathbf{x}_t, t)\}$ is called the systematic part of the $q$th derivative, ${}^{(q)}B_{\underbrace{1\dots1}_{q}}\{R(\mathbf{x}_t, t)\}$ is called the diffusion part of the $q$th derivative, and the remaining processes ${}^{(r_q)}B_{\gamma_q\dots\gamma_1}\{R(\mathbf{x}_t, t)\}$ are called mixed parts of the $q$th derivative. In the case $q = 0$, we set $r_q \equiv 0$; ${}^{(r_q)}B_{\gamma_q\dots\gamma_1}\{\cdot\} \overset{\text{def}}{=} \cdot$; ${}^{(r_q)}B_{0\gamma_q\dots\gamma_1}\{\cdot\} \overset{\text{def}}{=} {}^{(0)}B_0\{\cdot\}$; ${}^{(r_q+1)}B_{1\gamma_q\dots\gamma_1}\{\cdot\} \overset{\text{def}}{=} {}^{(1)}B_1\{\cdot\}$.*

Let ${}^{(1)}D_j\{\cdot\} = \left\|D_j^{(i)}\{\cdot\}\right\|_{i=1}^{m}$, $j = 1, \dots, k$, be vector differential operators and let

$$A_p\{\cdot\} = \begin{cases} C_p\{\cdot\} \\ \cdot \end{cases}, \quad p = 1, \dots, k+1,$$

where $C_p\{\cdot\}$ is a scalar differential operator.

Denote

$$\left\| A_{k+1}\left\{D_k^{(i_k)}\left\{A_k\left\{\dots\left\{D_1^{(i_1)}\left\{A_1\left\{\cdot\right\}\right\}\right\}\dots\right\}\right\}\right\|_{i_1,\dots i_k=1}^{m}$$

$$\stackrel{\text{def}}{=} {}^{(k)}A_{k+1}D_kA_k\dots D_1A_1\{\cdot\} = A_{k+1}\left\{{}^{(k)}D_kA_k\dots D_1A_1\{\cdot\}\right\}$$

$$= A_{k+1}\left\{{}^{(1)}D_k\left\{{}^{(k-1)}A_k\dots D_1A_1\{\cdot\}\right\}\right\}$$

$$\dots = A_{k+1}\left\{{}^{(1)}D_k\left\{A_k\dots\left\{{}^{(1)}D_1\left\{A_1\{\cdot\}\right\}\dots\right\}\right\}\right\}.$$

**Lemma 2.** *Assume that all the conditions of Lemma 1 are satisfied. Assume that the processes* ${}^{(r_l+1)}H_{l+1}H_l\dots H_1\{R(\mathbf{x}_s,s)\}$ *and the functions* ${}^{(r_l+1)}H_{l+1}H_l\dots H_1\{R(\mathbf{x},s)\}$ *satisfy conditions* (1°) *and* (2°) *of Lemma 1 componentwise for all* $l=0,1,\dots,N-2$. *Then the process* $\eta_s=R(\mathbf{x}_s,s)$ *is* $N$*-times Ito continuously differentiable in the mean-square sense on* $[0,T]$, *and its derivatives are given by the following formulas:*

$${}^{(r_l+1)}B_{\gamma_{l+1}\dots\gamma_1}\{R(\mathbf{x}_s,s)\} = {}^{(r_l+1)}H_{l+1}H_l\dots H_1\{R(\mathbf{x}_s,s)\},$$

*where* $H_p\{\cdot\}={}^{(1)}G_0\{\cdot\}, L\{\cdot\}$; $r_{l+1}=\sum\limits_{p=1}^{l+1}\gamma_p$; $\gamma_p=1$ *for* $H_p\{\cdot\}={}^{(1)}G_0\{\cdot\}$ *and* $\gamma_p=0$ *for* $H_p\{\cdot\}=L\{\cdot\}$. *The differential operators* ${}^{(1)}G_0\{\cdot\}$ *and* $L\{\cdot\}$ *are the same as in Lemma 1;* $l=0,1,\dots,N-1$.

## 3. Repeated stochastic integrals and their properties

Let us mention some properties of repeated stochastic integrals that we need below. One can find proofs of the corresponding statements in [9, 11].

Following (2) and (3), consider a repeated stochastic integral of the form

$$J_{ab}^{(k)} \stackrel{\text{def}}{=} \int_a^b \psi_{k-1}(t_1)\dots\int_a^{t_{k-2}}\psi_1(t_{k-1})\int_a^{t_{k-1}}\phi_{t_k}dW_{t_k}^{(i_k)}dW_{t_{k-1}}^{(i_{k-1})}\dots dW_{t_1}^{(i_1)},$$

where $\psi_j(t)$ $j=1,\dots,k-1$, are some functions, $\phi_t$ is a random process, and $W_t^{(q)}=\mathbf{f}_t^{(q)}$ for $q=1,\dots,m$ and $W_t^{(q)}=t$ for $q=0$, where $\mathbf{f}_t^{(q)}$ are independent scalar Wiener processes.

Let us formulate sufficient conditions for the existence of repeated stochastic integrals $J_{ab}^{(k)}$ in the mean-square sense [9, 11].

**Lemma 3.** *Let the functions* $\psi_j(t)$, $j=1,\dots,k-1$, *be continuous on* $[a,b]$. *Assume that the process* $\phi_t$ *is adapted with respect to the Wiener processes* $\mathbf{f}_t^{(q)}$ $(q=1,\dots,m)$ *and continuous in the mean square sense on* $[a,b]$. *Assume, in addition, that* $M\left\{\phi_t^2\right\}<\infty$ *for all* $t\in[a,b]$. *Then the repeated stochastic integrals* $J_{ab}^{(k)}$ $k=1,2,\dots$, *exist in the mean-square sense.*

Consider the property of change of integration order in repeated stochastic integrals.

**Definition 3.** *The mean-square limit*

$$\underset{\Delta_N\to 0}{\text{l.i.m.}}\sum_{j=0}^{N-1}\phi_{\tau_j}\left(W_{\tau_{j+1}}^{(i_k)}-W_{\tau_j}^{(i_k)}\right)S_{\tau_{j+1}b}^{(k-1)},$$

*where*

$$S_{t_1b}^{(k-1)} \stackrel{\text{def}}{=} \begin{cases} \int_{t_1}^b\psi_1(t_2)dW_{t_2}^{(i_{k-1})}\dots\int_{t_{k-1}}^b\psi_{k-1}(t_k)dW_{t_k}^{(i_1)} & \text{for } k>1,\\ 1 & \text{for } k=1,\end{cases}$$

*is called the integral*

$$I_{ab}^{(k)} \stackrel{\text{def}}{=} \int_a^b\phi_{t_1}dW_{t_1}^{(i_k)}\int_{t_1}^b\psi_1(t_2)dW_{t_2}^{(i_{k-1})}\dots\int_{t_{k-1}}^b\psi_{k-1}(t_k)dW_{t_k}^{(i_1)}.$$

The existence of the integral $I_{ab}^{(k)}$ and the property of change of integration order are established by the following theorem.

**Theorem 1.** *Assume that the functions $\psi_l(t)$, $l = 1, \dots, k-1$, and the process $\phi_t$ satisfy the conditions of Lemma 3. Then the integral $I_{ab}^{(k)}$ exists in the mean-square sense, and the equality*

$$I_{ab}^{(k)} = J_{ab}^{(k)}$$

*holds with probability* 1.

The following statement is a corollary of Theorem 1.

**Corollary.** *Under the conditions of Theorem* 1, *the equality*

$$\int_a^b \int_a^{t_1} \phi_\tau dW_\tau^{(j)} dW_{t_1}^{(i_k)} S_{t_1 b}^{(k-1)} = \int_a^b \phi_\tau dW_\tau^{(j)} \int_\tau^b dW_{t_1}^{(i_k)} S_{t_1 b}^{(k-1)}$$

*holds with probability* 1 *for* $j = 0, \dots, m$.

*Proof.* Consider the process $F_{at} = \int_a^t \phi_\tau dW_\tau^{(j)}$. Under the conditions of Theorem 1, the equality

$$\int_a^b F_{at_1} dW_{t_1}^{(i_k)} S_{t_1 b}^{(k-1)} = \int_a^b \psi_{k-1}(t_1) \dots \int_a^{t_{k-2}} \psi_1(t_{k-1}) \int_a^{t_{k-1}} F_{at_k} dW_{t_k}^{(i_k)} dW_{t_{k-1}}^{(i_{k-1})} \dots dW_{t_1}^{(i_1)}$$

holds. It remains to apply Theorem 1 to the right-hand side of the latter equality.

Below we use the following property of repeated stochastic integrals.

**Lemma 4.** *Assume that the conditions of Theorem* 1 *are fulfilled. Let $h(t)$ be a continuous function on $[a,b]$. Then the equality*

$$\int_a^b \phi_t dW_t^{(i_k)} h(t) S_{tb}^{(k-1)} = \int_a^b \phi_t h(t) dW_t^{(i_k)} S_{tb}^{(k-1)}$$

*holds with probability 1 for $k > 1$, and the integrals $\int_a^b \phi_t dW_t^{(i_k)} h(t) S_{tb}^{(k-1)}$ and $\int_a^b \phi_t h(t) dW_t^{(i_k)} S_{tb}^{(k-1)}$ exist in the mean-square sense.*

The following properties of the integrals $J_{ab}^{(k)}$ hold under the conditions of Lemma 3:

$$M\{J_{ab}^{(k)}\} = 0, \tag{8}$$

$$M\{\left(J_{ab}^{(k)}\right)^2\} = \sigma_{f_{i_1}}^2 \sigma_{f_{i_2}}^2 \dots \sigma_{f_{i_k}}^2 \int_a^b \psi_{k-1}^2(t_1) \dots \int_a^{t_{k-2}} \psi_1^2(t_{k-1}) \int_a^{t_{k-1}} M\{\phi_{t_k}^2\} dt_k dt_{k-1} \dots dt_1. \tag{9}$$

Introduce the following notation:

$$J_{l_1 \dots l_{k s,t}}^{(i_1 \dots i_k)} = \begin{cases} \int\limits_t^s (s-\tau_1)^{l_1} d\mathrm{f}_{\tau_1}^{(i_1)} \int\limits_{\tau_1}^s (s-\tau_2)^{l_2} d\mathrm{f}_{\tau_2}^{(i_2)} \dots \int\limits_{\tau_{k-1}}^s (s-\tau_k)^{l_k} d\mathrm{f}_{\tau_k}^{(i_k)}, & k > 0, \\ 1, & k = 0, \end{cases}$$

and

$$^{(k)}J_{l_1 \dots l_{k s,t}} = \left\| J_{l_1 \dots l_{k s,t}}^{(i_1 \dots i_k)} \right\|_{i_1, \dots, i_k = 1}^m .$$

Theorem 1 and properties (8) and (9) imply the existence and the following properties of the integrals $J_{l_1 \dots l_{k s,t}}^{(i_1 \dots i_k)}$ for $k > 0$:

$$M\{J_{l_1 \dots l_{k s,t}}^{(i_1 \dots i_k)}\} = 0;$$

$$M\left\{\left(J_{l_1 \dots l_{k s,t}}^{(i_1 \dots i_k)}\right)^2\right\}$$

$$= \frac{\sigma^2_{f_{i_1}} \sigma^2_{f_{i_2}} \dots \sigma^2_{f_{i_k}} (s-t)^{2(l_1+\dots+l_k)+k}}{(2l_k+1)(2(l_k+l_{k-1})+2)\dots(2(l_k+\dots+l_1)+k)}; \tag{10}$$

$$\int_t^s (s-\tau)^j d\mathrm{f}_\tau^{(p)} J^{(i_1\dots i_k)}_{l_1\dots l_{k s,\tau}} = J^{(pi_1\dots i_k)}_{jl_1\dots l_{k s,t}}, \quad p=1,\dots,m; \tag{11}$$

$$\int_t^s (s-\tau)^j d\tau J^{(i_1\dots i_k)}_{l_1\dots l_{k s,\tau}} = \frac{(s-t)^{j+1}}{j+1} J^{(i_1\dots i_k)}_{l_1\dots l_{k s,t}} - \frac{1}{j+1} J^{(i_1\dots i_k)}_{l_1+j+1 l_2\dots l_{k s,t}}; \tag{12}$$

$$J^{(i_1\dots i_k)}_{l_1\dots l_{k s,t}} = \int_t^s (s-\tau_1)^{l_k} \dots \int_t^{\tau_{k-1}} (s-\tau_k)^{l_1} d\mathrm{f}^{(i_1)}_{\tau_k} \dots d\mathrm{f}^{(i_k)}_{\tau_1}.$$

Introduce a matrix of rank $k$ of the form

$$^{(k)}(s \ominus t)_{jl_1\dots l_k} = \left\| (s\ominus t)^{(i_1\dots i_k)}_{jl_1\dots l_k} \right\|^m_{i_1,\dots,i_k=1} = \left\| \frac{(s-t)^j}{j!} J^{(i_1\dots i_k)}_{l_1\dots l_{k s,t}} \right\|^m_{i_1,\dots,i_k=1}$$

Consider the following family of matrices of rank $k$:

$$C^B = \left\{ ^{(k)}C_{jl_1\dots l_k} : (k,j,l_1,\dots,l_k) \in B \right\},$$

where

$$B = \{(k,j,l_1,\dots,l_k) : k,j,l_1,\dots,l_k \in Z_1 \subset \mathcal{N} \cup \{0\}\},$$

and define an operation on the set of matrices of rank $k$ by the formula

$$\left(C^B \odot D^B\right) \stackrel{\text{def}}{=} \sum_{(k,j,l_1,\dots,l_k)\in B} {}^{(k)}C_{jl_1\dots l_k} \overset{k}{\vdots} {}^{(k)}D_{jl_1\dots l_k}.$$

It is easy to see that the operation defined in this way is commutative and distributive.

## 4. Expansion of Ito processes

Let us prove a theorem on expansion for a process $\eta_s = R(\mathbf{x}_s, s)$.

**Theorem 2.** *Assume that an Ito process $\eta_s = R(\mathbf{x}_s, s)$ generated by a solution of Eq. (1) is $(r+1)$-times Ito continuously differentiable in the mean-square sense on $[0,T]$ along trajectories of Eq. (1). Then this process can be represented by the series*

$$\eta_s = \sum_{q=0}^{r} \left(C^{A_q}\{\eta_t\} \odot (s\ominus t)^{A_q}\right) + D_{r+1_{s,t}} \tag{13}$$

*in a neighborhood of a fixed time $t \in [0,T]$ so that the equality in (13) holds with probability 1, the right-hand side of (13) exists in the mean-square sense, and the following notation is adopted:*

$$D_{r+1_{s,t}} = \int_t^s \left(Q^{A_r}\{\eta_\tau\} d\tau \odot (s\ominus\tau)^{A_r}\right) + \int_t^s \left(\left(H^{A_r}\{\eta_\tau\} \overset{1}{\vdots} d\mathrm{f}_\tau\right) \odot (s\ominus\tau)^{A_r}\right), \tag{14}$$

*where*

$$C^{A_q}\{\eta_\tau\} = \left\{ ^{(k)}C_{jl_1\dots l_k}\{\eta_t\} : (j,l_1,\dots,l_k) \in A_q \right\};$$

$$Q^{A_q}\{\eta_\tau\} = \left\{ L\left\{ ^{(k)}C_{jl_1\dots l_k}\{\eta_\tau\}\right\} : (j,l_1,\dots,l_k) \in A_q \right\};$$

$$H^{A_q}\{\eta_\tau\} = \left\{ ^{(1)}G_0\left\{ ^{(k)}C_{jl_1\dots l_k}\{\eta_\tau\}\right\} : (j,l_1,\dots,l_k) \in A_q \right\};$$

$$
{}^{(k)}C_{jl_1\ldots l_k}\{\eta_t\} = \begin{cases} {}^{(k)}L^jG_{l_1}\ldots G_{l_k}\{\eta_t\} & \text{for } k>0 \\ L^j\{\eta_t\} & \text{for } k=0 \end{cases};
$$

$$
A_q = \{(k,j,l_1,\ldots,l_k) : k+j+l_1+\ldots+l_k = q; k,j,l_1,\ldots,l_k = 0,1,\ldots\};
$$

$$
L^j\{\cdot\} \stackrel{\text{def}}{=} \begin{cases} \underbrace{L\{L\{\ldots\{L}_{j}\{\cdot\}\}\ldots\}\} & \text{for } j>0 \\ \cdot & \text{for } j=0 \end{cases};
$$

$$
{}^{(1)}G_p\{\cdot\} = \frac{1}{p}\left({}^{(1)}G_{p-1}L\{\cdot\} - {}^{(1)}LG_{p-1}\{\cdot\}\right), \quad p=1,2,\ldots;
$$

${}^{(1)}G_0\{\cdot\}$ *and* $L\{\cdot\}$ *are defined by relations* (6) *and* (7), *respectively.*

*Proof.* We use induction to prove our statement. Apply the Ito formula to the process $\eta_s$:

$$
\eta_s = \eta_t + D_{1_{s,t}} = \left(C^{A_0}\{\eta_t\}\odot(s\ominus t)^{A_0}\right) + D_{1_{s,t}}, \tag{15}
$$

where

$$
D_{1_{s,t}} = \int_t^s L\{\eta_\tau\}d\tau + \int_t^s G_0\{\eta_\tau\}\,!\,d\mathbf{f}_\tau
$$

$$
= \int_t^s \left(Q^{A_0}\{\eta_\tau\}d\tau\odot(s\ominus\tau)^{A_0}\right) + \int_t^s \left(\left(H^{A_0}\{\eta_\tau\}\,!\,d\mathbf{f}_\tau\right)\odot(s\ominus\tau)^{A_0}\right). \tag{16}
$$

Relations (15) and (16) are particular cases of (13) and (14) for $r=0$. Apply the Ito formula to the integrands in $D_{1_{s,t}}$:

$$
D_{1_{s,t}} = D^*_{1_{s,t}} + D_{2_{s,t}}, \tag{17}
$$

where

$$
D^*_{1_{s,t}} = (s-t)L\{\eta_t\} + G_0\{\eta_t\}\,!\,J_{0_{s,t}} = \left(C^{A_1}\{\eta_t\}\odot(s\ominus t)^{A_1}\right), \tag{18}
$$

$$
D_{2_{s,t}} = \int_t^s\left(\int_t^{s_1} L^2\{\eta_\tau\}d\tau + \int_t^{s_1} G_0L\{\eta_\tau\}\,!\,d\mathbf{f}_\tau\right)ds_1
$$

$$
+\int_t^s\left(\int_t^{s_1} LG_0\{\eta_\tau\}d\tau + \int_t^{s_1} {}^{(2)}G_0G_0\{\eta_\tau\}\,!\,d\mathbf{f}_\tau\right)\,!\,d\mathbf{f}_{s_1}. \tag{19}
$$

Changing the order of integration in (19) by Theorem 1, we obtain the following relations:

$$
D_{2_{s,t}} = \int_t^s L^2\{\eta_\tau\}d\tau(s-\tau) + \int_t^s LG_0\{\eta_\tau\}d\tau\,!\,J_{0_{s,\tau}}
$$

$$
+\int_t^s G_0L\{\eta_\tau\}\,!\,d\mathbf{f}_\tau(s-\tau) + \int_t^s\left({}^{(2)}G_0G_0\{\eta_\tau\}\,!\,d\mathbf{f}_\tau\right)\,!\,J_{0_{s,\tau}}
$$

$$
= \int_t^s \left(Q^{A_1}\{\eta_\tau\}d\tau\odot(s\ominus\tau)^{A_1}\right) + \int_t^s \left(\left(H^{A_1}\{\eta_\tau\}\,!\,d\mathbf{f}_\tau\right)\odot(s\ominus\tau)^{A_1}\right). \tag{20}
$$

Relations (15)–(20) lead to the representation

$$
\eta_s = \sum_{q=0}^{1}\left(C^{A_q}\{\eta_t\}\odot(s\ominus t)^{A_q}\right) + D_{2_{s,t}}, \tag{21}
$$

where

$$
D_{2_{s,t}} = \int_t^s \left(Q^{A_1}\{\eta_\tau\}d\tau\odot(s\ominus\tau)^{A_1}\right) + \int_t^s \left(\left(H^{A_1}\{\eta_\tau\}\,!\,d\mathbf{f}_\tau\right)\odot(s\ominus\tau)^{A_1}\right). \tag{22}
$$

It is easy to see that relations (21) and (22) are particular cases of (13) and (14) for $r = 1$. Thus, our theorem is proved for $r = 0, 1$. Continuing the described expansion process, at the next step we get the following relations:

$$\eta_s = \sum_{q=0}^{1} \left(C^{A_q}\{\eta_t\} \odot (s \ominus t)^{A_q}\right) + \frac{1}{2}(s-t)^2 L^2\{\eta_t\}$$

$$+ \left(G_0 L\{\eta_t\} - LG_0\{\eta_t\}\right) \overset{1}{\cdot} J_{1_{s,t}} + {}^{(2)}G_0G_0\{\eta_t\} \overset{2}{\cdot} {}^{(2)}J_{00_{s,t}}$$

$$+ (s-t)LG_0\{\eta_t\} \overset{1}{\cdot} J_{0_{s,t}} + D_{3_{s,t}}$$

$$= \sum_{q=0}^{1} \left(C^{A_q}\{\eta_t\} \odot (s \ominus t)^{A_q}\right) + \left(C^{A_2}\{\eta_t\} \odot (s \ominus t)^{A_2}\right) + D_{3_{s,t}}$$

$$= \sum_{q=0}^{2} \left(C^{A_q}\{\eta_t\} \odot (s \ominus t)^{A_q}\right) + D_{3_{s,t}}, \tag{23}$$

where

$$D_{3_{s,t}} = \int_t^s L^3\{\eta_\tau\}\, d\tau \frac{1}{2}(s-\tau)^2 + \int_t^s L^2 G_0\{\eta_\tau\}\, d\tau \overset{1}{\cdot} J_{0_{s,\tau}}(s-\tau)$$

$$+ \int_t^s \left(LG_0L\{\eta_\tau\} - L^2G_0\{\eta_\tau\}\right) d\tau \overset{1}{\cdot} J_{1_{s,\tau}}$$

$$+ \int_t^s {}^{(2)}LG_0G_0\{\eta_\tau\}\, d\tau \overset{2}{\cdot} {}^{(2)}J_{00_{s,\tau}} + \int_t^s \left(G_0L^2\{\eta_\tau\} \overset{1}{\cdot} d\mathbf{f}_\tau\right) \frac{1}{2}(s-\tau)^2$$

$$+ \int_t^s \left({}^{(2)}G_0LG_0\{\eta_\tau\} \overset{1}{\cdot} d\mathbf{f}_\tau\right) \overset{1}{\cdot} J_{0_{s,\tau}}(s-\tau)$$

$$+ \int_t^s \left(\left({}^{(2)}G_0G_0L\{\eta_\tau\} - {}^{(2)}G_0LG_0\{\eta_\tau\}\right) \overset{1}{\cdot} d\mathbf{f}_\tau\right) \overset{1}{\cdot} J_{1_{s,\tau}}$$

$$+ \int_t^s \left({}^{(3)}G_0G_0G_0\{\eta_\tau\} \overset{1}{\cdot} d\mathbf{f}_\tau\right) \overset{2}{\cdot} {}^{(2)}J_{00_{s,\tau}}$$

$$= \int_t^s \left(Q^{A_2}\{\eta_\tau\} d\tau \odot (s \ominus \tau)^{A_2}\right) + \int_t^s \left(\left(H^{A_2}\{\eta_\tau\} \overset{1}{\cdot} d\mathbf{f}_\tau\right) \odot (s \ominus \tau)^{A_2}\right). \tag{24}$$

Hence, the statement of our theorem is valid for $r = 2$. Assume that the statement of our theorem is valid for some $n = r$. Let us prove that it is then valid for $n = r + 1$. Applying the Ito formula to the integrands in (14), we get the relations

$$D_{r+1_{s,t}} = D^*_{r+1_{s,t}} + D_{r+2_{s,t}},$$

where

$$D^*_{r+1_{s,t}} = \left(Q^{A_r}\{\eta_t\} \odot \int_t^s d\tau (s \ominus \tau)^{A_r}\right) + \left(\left(H^{A_r}\{\eta_t\} \overset{1}{\cdot} \int_t^s d\mathbf{f}_\tau\right) \odot (s \ominus \tau)^{A_r}\right),$$

$$D_{r+2_{s,t}} = \int_t^s \left(\int_t^{\tau_1} dU^{A_r}\{\eta_\tau\} d\tau_1 \odot (s \ominus \tau_1)^{A_r}\right) + \int_t^s \left(\left(\int_t^{\tau_1} dV^{A_r}\{\eta_\tau\} \overset{1}{\cdot} d\mathbf{f}_{\tau_1}\right) \odot (s \ominus \tau_1)^{A_r}\right), \tag{25}$$

and

$$dU^{A_r}\{\eta_\tau\} = \left\{{}^{(k)}X_{jl_1\ldots l_k}\{\eta_\tau\} d\tau + {}^{(k+1)}Y_{jl_1\ldots l_k}\{\eta_\tau\} \overset{1}{\cdot} d\mathbf{f}_\tau : (j, l_1, \ldots, l_k, k) \in A_r\right\},$$

$$dV^{A_r}\{\eta_\tau\} = \left\{{}^{(k+1)}Z_{jl_1\ldots l_k}\{\eta_\tau\} d\tau + {}^{(k+2)}W_{jl_1\ldots l_k}\{\eta_\tau\} \overset{1}{\cdot} d\mathbf{f}_\tau : (j, l_1, \ldots, l_k, k) \in A_r\right\},$$

$$ {}^{(k)}X_{jl_1\dots l_k}\{\eta_\tau\} = {}^{(k)}L^{j+2}G_{l_1}\dots G_{l_k}\{\eta_\tau\}, $$

$$ {}^{(k+1)}Z_{jl_1\dots l_k}\{\eta_\tau\} = {}^{(k+1)}LG_0L^jG_{l_1}\dots G_{l_k}\{\eta_\tau\}, $$

$$ {}^{(k+1)}Y_{jl_1\dots l_{k_\tau}} = {}^{(k+1)}G_0L^{j+1}G_{l_1}\dots G_{l_k}\{\eta_\tau\}, $$

$$ {}^{(k+2)}W_{jl_1\dots l_{k_\tau}} = {}^{(k+2)}G_0G_0L^jG_{l_1}\dots G_{l_k}\{\eta_\tau\}. $$

By the corollary of Theorem 1, it is possible to change the order of integration in repeated stochastic integrals in (25), hence we have

$$ D_{r+2_{s,t}} = \int_t^s \left(\left(dU^{A_r}\{\eta_\tau\}\int_\tau^s d\tau_1\right)\odot(s\ominus\tau_1)^{A_r}\right) + \int_t^s \left(\left(dV^{A_r}\{\eta_\tau\}!\int_\tau^s d\mathrm{f}_{\tau_1}\right)\odot(s\ominus\tau_1)^{A_r}\right). \quad (26) $$

Consider the integrals $\int_\tau^s d\tau_1(s\ominus\tau_1)^{(i_1\dots i_k)}_{jl_1\dots l_k}$ and $\int_\tau^s d\mathrm{f}^{(q)}_{\tau_1}(s\ominus\tau_1)^{(i_1\dots i_k)}_{jl_1\dots l_k}d\tau_1$, $q=1,\dots,m$.

By properties (11) and (12), we have

$$ \int_\tau^s d\tau_1(s\ominus\tau_1)^{(i_1\dots i_k)}_{jl_1\dots l_k} $$

$$ = \begin{cases} (s\ominus\tau)^{(i_1\dots i_k)}_{j+1l_1\dots l_k} - \frac{1}{(s-\tau)^{j+1}}(s\ominus\tau)^{(i_1\dots i_k)}_{j+1,l_1+j+1,l_2\dots l_k} & \text{for } k>0 \\ \frac{(s-\tau)^{j+1}}{(j+1)!} & \text{for } k=0 \end{cases}; \quad (27) $$

$$ \int_\tau^s d\mathrm{f}^{(q)}_{\tau_1}(s\ominus\tau_1)^{(i_1\dots i_k)}_{jl_1\dots l_k} = \frac{1}{(s-\tau)^j}(s\ominus\tau)^{(qi_1\dots i_k)}_{jjl_1\dots l_k}, \quad k=0,1,\dots. \quad (28) $$

Substitute (27) and (28) into (26) and note that the result obtained is equivalent to the following relation:

$$ D_{r+2_{s,t}} = \int_t^s \left(Q^{A_{r+1}}\{\eta_\tau\}d\tau\odot(s\ominus\tau)^{A_{r+1}}\right) + \int_t^s \left(\left(H^{A_{r+1}}\{\eta_\tau\}!d\mathrm{f}_\tau\right)\odot(s\ominus\tau)^{A_{r+1}}\right). $$

This proves (14). It follows from (14) that

$$ D_{r+1_{s,t}} = D^*_{r+1_{s,t}} + D_{r+2_{s,t}}, $$

where

$$ D^*_{r+1_{s,t}} = \left(Q^{A_r}\{\eta_t\}\int_t^s d\tau\odot(s\ominus\tau)^{A_r}\right) + \left(\left(H^{A_r}\{\eta_t\}!\int_t^s d\mathrm{f}_\tau\right)\odot(s\ominus\tau)^{A_r}\right). \quad (29) $$

Substituting (27) and (28) with $\tau=t$ into (29), we arrive at the formula

$$ D^*_{r+1_{s,t}} = \left(C^{A_{r+1}}\{\eta_t\}\odot(s\ominus t)^{A_{r+1}}\right). $$

This proves relation (13). Our theorem is proved.

## 5. Unified Taylor–Ito series

It is easy to see that some terms on the right-hand side of formula (13) have higher orders of smallness in the mean-square sense as $s\to t$ than the remainder term $D_{r+1_{s,t}}$. This conclusion follows from property (10). The following theorem describes a modification of expansion (13) such that the terms of the modified expansion are ordered according to their orders of smallness. In this case, the remainder term has the highest order of smallness.

**Theorem 3.** *Under the conditions of Theorem 2, the following unified Taylor–Ito expansion of a process $\eta_s = R(\mathbf{x}_s, s)$, $s \in [0,T]$, holds in a neighborhood of a fixed time $t \in [0,T]$:*

$$\eta_s = \sum_{q=0}^{r} \left(C^{D_q}\{\eta_t\} \odot (s \ominus t)^{D_q}\right) + H_{r+1_{s,t}},$$

*where*

$$H_{r+1_{s,t}} \stackrel{\text{def}}{=} \left(C\{\eta_t\}^{U_r} \odot (s \ominus t)^{U_r}\right) + D_{r+1_{s,t}},$$

$$D_{r+1_{s,t}} = \int_t^s \left(Q^{A_q}\{\eta_\tau\}d\tau \odot (s \ominus \tau)^{A_r}\right) + \int_t^s \left(\left(H^{A_q}\{\eta_\tau\}^! d\mathbf{f}_\tau\right) \odot (s \ominus \tau)^{A_r}\right),$$

$$D_q = \{(k,j,l_1,\dots,l_k) : k + 2(j + l_1 + \ldots + l_k) = q; \;\; k,j,l_1,\dots,l_k = 0,1,\dots\},$$

$$U_r = \{(k,j,l_1,\dots,l_k) : k + j + l_1 + \ldots + l_k \le r;$$

$$k + 2(j + l_1 + \ldots + l_k) \ge r+1; k,j,l_1,\dots,l_k = 0,1,\dots\},$$

$$\sqrt{M\left\{(H_{r+1_{s,t}})^2\right\}} \le C_r(s-t)^{(r+1)/2}; \; C_r = \text{const} < \infty; r = 0,1,\dots,$$

*and the remaining notation coincides with that in Theorem 2.*

*Proof.* It is easy to see that it is possible to represent expansion (13) in the following form:

$$\eta_s = \sum_{q=0}^{r} \left(C^{D_q}\{\eta_t\} \odot (s \ominus t)^{D_q}\right) + H_{r+1_{s,t}},$$

where

$$H_{r+1_{s,t}} \stackrel{\text{def}}{=} \left(C^{U_r}\{\eta_t\} \odot (s \ominus t)^{U_r}\right) + D_{r+1_{s,t}}.$$

It follows from the Minkowski inequality that

$$\sqrt{M\left\{(H_{r+1_{s,t}})^2)\right\}} \le \sqrt{M\left\{\left(C^{U_r}\{\eta_t\} \odot (s \ominus t)^{U_r}\right)^2\right\}} + \sqrt{M\left\{(D_{r+1_{s,t}})^2\right\}}. \tag{30}$$

Let us estimate the terms of the right-hand side of (30). Since

$$\left(C^{U_r}\{\eta_t\} \odot (s \ominus t)^{U_r}\right)$$

$$= \sum_{(k,j,l_1,\dots,l_k) \in U_r} \frac{(s-t)^j}{j!} \sum_{i_1,\dots,i_k=1}^{m} L^j G_{l_1}^{(i_1)} \ldots G_{l_k}^{(i_k)} \{\eta_t\} J_{l_1 \ldots l_k s,t}^{(i_1 \ldots i_k)},$$

it follows from property (10) and from the Minkowski inequality that, in a small neighborhood of time $t$, the inequality

$$\sqrt{M\left\{\left(C^{U_r}\{\eta_t\} \odot (s \ominus t)^{U_r}\right)^2\right\}} \le C_r'(s-t)^z, \quad C_r' = \text{const} < \infty,$$

holds, where

$$z = \min_{(k,j,l_1,\dots,l_k) \in U_r} \left\{\frac{k}{2} + j + l_1 + \ldots + l_k\right\} = \frac{r+1}{2}.$$

Thus, we have the inequality

$$\sqrt{M\left\{\left(C\{\eta_t\}^{U_r} \odot (s \ominus t)^{U_r}\right)^2\right\}} \le C_r'(s-t)^{\frac{r+1}{2}}, \quad C_r' = \text{const} \; < \infty. \tag{31}$$

Consider the value $D_{r+1_{s,t}}$:

$$D_{r+1_{s,t}} = \int_t^s \left(Q^{A_r}\{\eta_\tau\}d\tau \odot (s\ominus\tau)^{A_r}\right) + \int_t^s \left(\left(H^{A_r}\{\eta_\tau\} \stackrel{1}{\cdot} d\mathbf{f}_\tau\right) \odot (s\ominus\tau)^{A_r}\right)$$

$$= \sum_{(k,j,l_1,\ldots,l_k)\in A_r} \sum_{i_1,\ldots,i_k=1}^{m} \left( \int_t^s L^{j+1}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}\{\eta_\tau\}\frac{(s-\tau)^j}{j!}d\tau J_{l_1\ldots l_{k\,s,\tau}}^{(i_1\ldots i_k)} \right.$$

$$\left. + \sum_{p=1}^{m}\int_t^s G_0^{(p)}L^jG_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}\{\eta_\tau\}\frac{(s-\tau)^j}{j!}d\mathbf{f}_\tau^{(p)}J_{l_1\ldots l_{k\,s,\tau}}^{(i_1\ldots i_k)}\right).$$

Theorem 1, property (10), and the Minkowski inequality imply that, in a small neighborhood of time $t$, the inequality

$$\sqrt{M\left\{(D_{r+1_{s,t}})^2\right\}} \le C_r^{''}(s-t)^{z'}, \quad C_r^{''} = \text{const} < \infty,$$

holds, where

$$z' = \min_{k+j+l_1+\ldots+l_k=r;0\le k\le r}\left\{1+\frac{k}{2}+j+l_1+\ldots+l_k;\ \frac{1}{2}+\frac{k}{2}+j+l_1+\ldots+l_k\right\} = \frac{r+1}{2}.$$

Thus, we have

$$\sqrt{M\left\{(D_{r+1_{s,t}})^2\right\}} \le C_r^{''}(s-t)^{\frac{r+1}{2}}, \quad C_r^{''} = \text{const} < \infty. \tag{32}$$

It follows from (30)–(32) that

$$\sqrt{M\left\{(H_{r+1_{s,t}})^2\right\}} \le C_r(s-t)^{\frac{r+1}{2}}, \quad C_r = \text{const} < \infty.$$

Our theorem is proved.

This research was supported by the Russian Foundation for Basic Research, grant 96-01-00929, and by the Program "Leading Scientific Schools," grant 96-15-96199.

Translated by S. Yu. Pilyugin.

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