A Hypothesis on the Expansion of Multiple Stratonovich Stochastic Integrals of Any Arbitrary Multiplicity k, Based on the Multiple Fourier Series Converging in the Mean-Square Sense

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Let $I_{T,t}^{*(i_1\ldots i_k)}$ be a multiple Stratonovich stochastic integral

$$\int\limits_t^{*T}\int\limits_t^{*t_k}\ldots\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}d\mathbf{w}_{t_k}^{(i_k)},$$

where $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ for $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$; $\mathbf{f}_\tau$ is a standard m-dimensional Wiener stochastic process with independent components $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$; $i_1,\ldots,i_k=0,\ 1,\ldots,m$.

Hypothesis 1 (see [1, p.A.155], [2, p.A.391-A.392]). *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in $L_2([t,T])$. Then, for multiple Stratonovich stochastic integral*

$$I_{T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}\int\limits_t^{*t_k}\ldots\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}d\mathbf{w}_{t_k}^{(i_k)}, \qquad (1)$$

the following converging in the mean-square sense expansion

$$I_{T,t}^{*(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots j_k=0}^{p} C_{j_k\ldots j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\ldots\zeta_{j_k}^{(i_k)} \qquad (2)$$

is reasonable, where

$$C_{j_k\ldots j_1}=\int\limits_t^{T}\phi_{j_k}(t_k)\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2\ldots dt_k;$$

$i_1,i_2,\ldots,i_k=0,1,\ldots,m$; *every*

$$\zeta_j^{(i)}=\int\limits_t^{T}\phi_j(s)d\mathbf{w}_s^{(i)}$$

is a standard Gaussian random variable for various i or j (if $i\neq 0$).

Note that (2) means the following

$$\lim_{p\to\infty}\mathsf{M}\left\{\left(I_{T,t}^{*(i_1\ldots i_k)}-\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\right\}=0,$$

where M denotes a mathematical expectaton.

Hypotisys 1 has been proven for the cases $k=1,\ 2,\ 3,\ 4$ in [1], [2].

See [1]: theorem 3 (p. A.59), theorem 4 (p. A.109-A.110), theorem 6 (p. A.116-A.117), theorem 6' (p. A.126), theorem 7 (p. A.129), theorem 8 (p. A.135), theorem 9 (p. A.155).

Also see [2]: theorem 5.3 (p. A.292), theorem 5.4 (p. A.345), theorem 5.5' (p. A.351-A.352), theorem 5.6 (p. A.363), theorem 5.7 (p. A.369-A.370), theorem 5.8 (p. A.391-A.392).

Note that in principle the Hypotisys 1 can be proven for the case $k\ge 5$ by the method considered in [1], [2].

Hypotisys 1 allows to approximate multiple Stratonovich stochastic integral $I_{T,t}^{*(i_1\ldots i_k)}$ by the sum

$$I_{T,t}^{*(i_1\ldots i_k)p}=\sum_{j_1,\ldots j_k=0}^{p}C_{j_k\ldots j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\ldots\zeta_{j_k}^{(i_k)}, \tag{3}$$

where

$$\lim_{p\to\infty}M\left\{\left(I_{T,t}^{*(i_1\ldots i_k)}-I_{T,t}^{*(i_1\ldots i_k)p}\right)^2\right\}=0.$$

Integrals (1) are integrals from the Taylor-Stratonovich expansion [3]. It means, that approximation (3) may be useful for numerical integration of Ito stochastic differential equations.

References

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[2] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 5th Ed. (In Russian). *Electronic Journal "Differential Equations and Control Processes"*, no. 2, 2017, 1000 (A.1 - A.1000) pp. (DOI: 10.18720/SPBPU/2/z17-4). Available at: http://www.math.spbu.ru/diffjournal/pdf/kuznetsov_book3.pdf

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