STRONG APPROXIMATION

OF MULTIPLE ITO AND

STRATONOVICH

STOCHASTIC INTEGRALS

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0. ITO STOCHASTIC DIFFERENTIAL EQUATIONS AND STOCHASTIC TAYLOR EXPANSIONS

1. $(\Omega, \mathrm{F}, \mathsf{P})$ — is a probability space;

2. $\mathbf{W}_t = \left\{\mathbf{W}_t^{(1)}, \ldots, \mathbf{W}_t^{(m)}\right\}$, $s \in [0, T]$ — is F_t-measurable for all $t \in [0, T]$ Wiener process with independent components;

3. $\mathbf{W}_t^{(0)} \stackrel{def}{=} t$ (time).

Let's analyze the Ito Stochastic differential equation (SDE):

$$\mathbf{X}_t = \mathbf{X}_0 + \int\limits_0^t A(\mathbf{X}_s, s)ds + \sum_{i=1}^m \int\limits_0^t B_i(\mathbf{X}_s, s)d\mathbf{W}_s^{(i)},\ \mathbf{X}_0 = \mathbf{X}(0, \omega), \tag{1}$$

where $A, B_1, \ldots, B_m : \Re^n \times [0, T] \to \Re^n$ satisfy the standard conditions of existence and uniqueness of strong solution $\mathbf{X}_t \in \Re^n$ of SDE (1); $\mathbf{X}_0$ and $\mathbf{W}_t - \mathbf{W}_0 (t > 0)$ — are independent.

From iterative application of Ito formula we have:

$$\mathbf{X}_s = \mathbf{X}_t + \sum_{i=1}^m B_i(\mathbf{X}_t, t) \int\limits_t^s d\mathbf{W}_\tau^{(i)} + A(\mathbf{X}_t, t) \int\limits_t^s d\tau +$$

$$+ \sum_{i,j=1}^m G_j B_i(\mathbf{X}_t, t) \int\limits_t^s \int\limits_t^\tau d\mathbf{W}_\theta^{(j)} d\mathbf{W}_\tau^{(i)} +$$

$$+ \sum_{i=1}^m \left(G_i A(\mathbf{X}_t, t) \int\limits_t^s \int\limits_t^\tau d\mathbf{W}_\theta^{(i)} d\tau + L B_i(\mathbf{X}_t, t) \int\limits_t^s \int\limits_t^\tau d\theta d\mathbf{W}_\tau^{(i)} \right) +$$

$$+ \sum_{i,j,k=1}^m G_k G_j B_i(\mathbf{X}_t, t) \int\limits_t^s \int\limits_t^\tau \int\limits_t^\theta d\mathbf{W}_u^{(k)} d\mathbf{W}_\theta^{(j)} d\mathbf{W}_\tau^{(i)} + \ldots \text{ w.p.1}, \tag{2}$$

where $s > t$ and L, G_i — are differential operators.

The expansion (2) can be rewritten in terms of multiple Stratonovich stochastic integrals.

So, our goal is mean-square approximation of multiple Ito and Stratonovich stochastic integrals in the form:

$$\int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{W}_{t_1}^{(i_1)} \ldots d\mathbf{W}_{t_k}^{(i_k)},$$

$$\int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{W}_{t_1}^{(i_1)} \ldots d\mathbf{W}_{t_k}^{(i_k)},$$

where

1. $\psi_1(\tau),\ \ldots, \psi_k(\tau)$ — are continuous or smooth functions at the interval $[t, T]$;

2. $\int\limits_t^T$ — is Ito stochastic integral;

3. $\int\limits_t^{*T}$ — is Stratonovich stochastic integral;

4. $i_1, \ldots, i_k = 0, 1, \ldots, m.$

1. EXPANSION OF

MULTIPLE ITO

STOCHASTIC INTEGRALS

(Main Theorem)

[1] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2 (In Russian). 2006, 764 pp. S.-Petersburg: Polytechnical University Press. ISBN: 5-7422-1191-0.

Available at (Free):

http://www.math.spbu.ru/diffjournal/pdf/kuznetsov_book3.pdf

http://www.math.spbu.ru/diffjournal/pdf/kuznetsov_book2.pdf

Denotations

1. $\psi_1(\tau), \ \ldots, \psi_k(\tau)$ — are continuous functions at $[t,T]$.

2.
$$K(t_1,\ldots,t_k)=$$

$$=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k),\ t<t_1<\ldots<t_k<T\\ \\ 0,\ \text{otherwise}\end{cases}\in L_2([t,T]^k).$$

3. $C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)dt_1\ldots dt_k$

— is a Fourier coefficient.

4. $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of continuous functions in the space $L_2([t,T])$.

5. $t=\tau_0<\ldots<\tau_N=T,\ \Delta_N=\max\limits_{0\le j\le N-1}|\tau_{j+1}-\tau_j|\to 0$ if $N\to\infty$.

6. $N_{ind}(0,1)$ — is an independent standard Gaussian random values.

7. l.i.m. — is a limit in mean-square.

Theorem 1 ([1], Kuznetsov D.F.) *Assume, that:*

1. $\psi_1(\tau),\ \ldots,\psi_k(\tau)$ — *are continuous functions at* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of continuous functions in the space* $L_2([t,T])$.

Then the multiple Ito stochastic integral

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}$$

is expanded in the multiple series converging in the mean-square sense

$$\int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}=$$

$$=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Bigl(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{W}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{W}_{\tau_{l_k}}^{(i_k)}\Bigr),$$

where $\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k$;

$$\mathrm{H}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\bigr\},$$

$$\mathrm{L}_k=\bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r(g\ne r);\ g,r=1,\ldots,k\bigr\};$$

$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{W}_s^{(i)}\sim N_{ind}(0,1)$ *for various* i *or* j *(if* $i\ne 0$*).*

Particular cases of the theorem 1 (k=1, 2, 3, 4):

$$\int_t^T \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)} = \sum_{j_1=0}^{\infty} C_{j_1}\zeta_{j_1}^{(i_1)},$$

$$\int_t^T \psi_2(t_2)\int_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}d\mathbf{W}_{t_2}^{(i_2)} = \sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}\Bigl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - $$

$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\Bigr),$$

$$\int_t^T \psi_3(t_3)\ldots\int_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_3}^{(i_3)} =$$

$$= \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\Bigl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Bigr),$$

$$\int_t^T \psi_4(t_4)\ldots\int_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_4}^{(i_4)} =$$

$$= \sum_{j_1,\ldots,j_4=0}^{\infty} C_{j_4\ldots j_1}\Bigl(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\neq 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_3\neq 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\neq 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\neq 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\neq 0\}}\mathbf{1}_{\{j_2=j_3\}}\Bigr).$$

2. EXPANSION OF MULTIPLE STRATONOVICH STOCHASTIC INTEGRALS (Theorems for integrals of 2-4 multiplicity)

[2] Dmitriy F. Kuznetsov. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Serieses (In Russian). Electronic Journal "Differential Equations and Control Processes" (ISSN 1817-2172). N3. 2010. 257 pp.

[3] Dmitriy F. Kuznetsov. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2-nd Edition (In English). 2011, 284 pp. S.-Petersburg: Polytechnical University Publishing House. ISBN: 978-5-7422-3162-2.

[4] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs. 5-th Edition. (In Russian). Electronic Journal "Differential Equations and Control Processes" (ISSN 1817-2172). N2. 2017. 1000 pp.

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http://www.math.spbu.ru/diffjournal/pdf/kuznetsov_book2.pdf

Theorem 2 ([2, 3, 4] Kuznetsov D.F.) *Assume, that the following conditions are met:*

1. *The function* $\psi_2(\tau)$ *is continuously differentiated at the interval* $[t,T]$ *and the function* $\psi_1(\tau)$ *is two times continuously differentiated at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of Legendre polynomials or system of trigonometric functions in the space* $L_2([t,T])$.

Then, the multiple Stratonovich stochastic integral of the second multiplicity is expanded into the converging in the mean-square sense multiple series

$$\int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{W}_{t_1}^{(i_1)} d\mathbf{W}_{t_2}^{(i_2)} =$$

$$= \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}$$

$(i_1, i_2 = 1, \ldots, m)$, *where the meaning of notations introduced in the formulations of theorem* 1 *is remained.*

Theorem 3 ([2, 3, 4] Kuznetsov D.F.) *Assume, that* $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of Legendre polynomials in the space* $L_2([t,T])$. *Then, for multiple Stratonovich stochastic integral of 3rd multiplicity the following converging in the mean-square sense expansion*

$$\int\limits_t^{*T}(t-t_3)^{l_3}\int\limits_t^{*t_3}(t-t_2)^{l_2}\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{W}_{t_1}^{(i_1)}d\mathbf{W}_{t_2}^{(i_2)}d\mathbf{W}_{t_3}^{(i_3)}=$$

$$=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

$(i_1,i_2,i_3=1,\ldots,m)$ *is reasonable for each of the following cases:*

1. $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3$ *and* $l_1,\ l_2,\ l_3=0,\ 1,\ 2,\ldots;$
2. $i_1=i_2\ne i_3$ *and* $l_1=l_2\ne l_3$ *and* $l_1,\ l_2,\ l_3=0,\ 1,\ 2,\ldots;$
3. $i_1\ne i_2=i_3$ *and* $l_1\ne l_2=l_3$ *and* $l_1,\ l_2,\ l_3=0,\ 1,\ 2,\ldots;$
4. $i_1,i_2,i_3=1,\ldots,m;\ l_1=l_2=l_3=l$ *and* $l=0,\ 1,\ 2,\ldots,$

where

$$C_{j_3j_2j_1}=$$

$$=\int\limits_t^T(t-s)^{l_3}\phi_{j_3}(s)\int\limits_t^s(t-s_1)^{l_2}\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^{l_1}\phi_{j_1}(s_2)ds_2ds_1ds.$$

In [2, 3, 4] (Kuznetsov D.F.) also the variant of case 4 ($i_1, i_2, i_3 = 1, \ldots, m$; $l_1 = l_2 = l_3 = 0$) from the theorem 3 for trigonometric functions is proven.

Theorem 4 ([3, 4] Kuznetsov D.F.) *Assume, that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *— is a full orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t, T])$, *function* $\psi_2(s)$ *— is continuously differentiated at the interval* $[t, T]$ *and functions* $\psi_1(s), \psi_3(s)$ *— are two times continuously differentiated at the interval* $[t, T]$.

Then, for multiple Stratonovich stochastic integral of 3*rd multiplicity the following converging in the mean-square sense expansion*

$$\int\limits_t^{*T} \psi_3(t_3) \int\limits_t^{*t_3} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{W}_{t_1}^{(i_1)} d\mathbf{W}_{t_2}^{(i_2)} d\mathbf{W}_{t_3}^{(i_3)} =$$

$$= \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}$$

$(i_1, i_2, i_3 = 1, \ldots, m)$ *is reasonable, where*

$$C_{j_3 j_2 j_1} =$$

$$= \int\limits_t^T \psi_3(s) \phi_{j_3}(s) \int\limits_t^s \psi_2(s_1) \phi_{j_2}(s_1) \int\limits_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 ds_1 ds.$$

Theorem 5 ([2, 3, 4] Kuznetsov D.F.). *Assume, that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *— is a full orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$.

Then, for multiple Stratonovich stochastic integral of 4*th multiplicity the following converging in the mean-square sense expansion*

$$\int\limits_t^{*T}\int\limits_t^{*t_4}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_4}^{(i_4)} =$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$

$(i_1,i_2,i_3,i_4=0,1,\ldots,m)$ *is reasonable, where*

$$C_{j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \phi_{j_4}(s)\int\limits_t^s \phi_{j_3}(s_1)\int\limits_t^{s_1} \phi_{j_2}(s_2)\int\limits_t^{s_2} \phi_{j_1}(s_3)ds_3ds_2ds_1ds;$$

3. EXACT AND ESTIMATE CALCULATION OF MEAN-SQUARE ERRORS OF APPROXIMATIONS BASED ON THEOREM 1

[4] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs. 5-th Edition. (In Russian). Electronic Journal "Differential Equations and Control Processes" (ISSN 1817-2172). N2. 2017. 1000 pp.

[5] Dmitriy F. Kuznetsov. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Triginometric Expansions, Approximations, Formulas. (In English). Electronic Journal "Differential Equations and Control Processes" (ISSN 1817-2172). N1. 2017. 385 pp.

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http://www.math.spbu.ru/diffjournal/pdf/kuznetsov_book2.pdf

Denotations

1. $\psi_1(\tau),\ \ldots,\psi_k(\tau)$ — *are continuous functions at* $[t,T]$.

2.
$$K(t_1,\ldots,t_k)=$$
$$=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k),\ t<t_1<\ldots<t_k<T\\ \\ 0,\ \text{otherwise}\end{cases}\in L_2([t,T]^k).$$

3. $C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)dt_1\ldots dt_k$

— *is a Fourier coefficient.*

4. $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of continuous functions in the space* $L_2([t,T])$.

5. $t=\tau_0<\ldots<\tau_N=T,\ \Delta_N=\max\limits_{0\le j\le N-1}|\tau_{j+1}-\tau_j|\to 0$ *if* $N\to\infty$.

6. $N_{ind}(0,1)$ — *is an independent standard Gaussian random values.*

7. $\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{W}_s^{(i)}\sim N_{ind}(0,1)$ *for various* i *or* j *(if* $i\neq 0$*).*

8. $\sum\limits_{(j_1,\ldots,j_k)}$ — *summation according to derangements.*

Theorem 6 ([4], Kuznetsov D.F.) *Assume, that:*

1. $\psi_1(\tau),\ \ldots,\psi_k(\tau)$ — *are continuous functions at* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of continuous functions in the space* $L_2([t,T])$.

3. $i_1,\ldots,i_k=1,\ldots,m.$

Then

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\times$$

$$\times\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)}\right\},$$

where

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)},$$

$$J[\psi^{(k)}]_{T,t}^{p}=\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\Bigl(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{W}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{W}_{\tau_{l_k}}^{(i_k)}\Bigr)$$

$($*approximation based on theorem* $1)$

Corrolary 1. *If* $i_1,\ldots,i_k$ *— are pairwise different, then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}^2.$$

Corrolary 2. *If* $i_1=\ldots=i_k$*, then*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{p}\ldots\sum_{j_k=0}^{p}C_{j_k\ldots j_1}\Big(\sum_{(j_1,\ldots,j_k)}C_{j_k\ldots j_1}\Big).$$

Case of 3rd multiplicity. Example $(i_1=i_2\neq i_3)$**.**

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^{p}-J[\psi^{(3)}]_{T,t}\right)^2\right\}=$$

$$=\int\limits_{[t,T]^3}K^2(t_1,t_2,t_3)dt_1dt_2dt_3-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_2j_1}^2-$$

$$-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}.$$

Case of 4th multiplicity. Example $(i_1 = i_2 \neq i_3 = i_4)$**.**

$$\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t} - J[\psi^{(4)}]_{T,t}^p\right)^2\right\} =$$

$$= \int\limits_{[t,T]^4} K^2(t_1,\ldots,t_4)dt_1\ldots dt_4 -$$

$$- \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\Bigl(\sum_{(j_3,j_4)}\Bigl(\sum_{(j_1,j_2)} C_{j_4j_3j_2j_1}\Bigr)\Bigr).$$

Case of 5th multiplicity. Example $(i_1 = i_2 = i_5 \neq i_3 = i_4)$.

$$\mathsf{M}\left\{\left(J[\psi^{(5)}]_{T,t} - J[\psi^{(5)}]_{T,t}^p\right)^2\right\} =$$

$$= \int\limits_{[t,T]^5} K^2(t_1,\ldots,t_5)dt_1\ldots dt_5 -$$

$$- \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\Bigl(\sum_{(j_3,j_4)}\Bigl(\sum_{(j_1,j_2,j_5)} C_{j_5\ldots j_1}\Bigr)\Bigr).$$

Mean-Square Estimates for the Cases $i_1,\ldots,i_k = 1,\ldots,m$ or $i_1,\ldots,i_k = 0,1,\ldots,m$ $(T-t<1)$.

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le
$$

$$
\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right),
$$

$$
\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2\right\}\le
$$

$$
\le k!\left(\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}^2\right),
$$

where

$$
J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Bigl(\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}-
$$

$$
-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{W}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{W}_{\tau_{l_k}}^{(i_k)}\Bigr)
$$

$(approximation\ based\ on\ theorem\ 1)$

4. MEAN-SQUARE APPROXIMATION OF CONCRETE MULTIPLE ITO AND STRATONOVICH STOCHASTIC INTEGRALS. CASES OF LEGENDRE POLYNOMIALS AND TRIGONOMETRIC FUNCTIONS

[4] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs. 5-th Edition. (In Russian). Electronic Journal "Differential Equations and Control Processes" (ISSN 1817-2172). N2. 2017. 1000 pp.

[5] Dmitriy F. Kuznetsov. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Triginometric Expansions, Approximations, Formulas. (In English). Electronic Journal "Differential Equations and Control Processes" (ISSN 1817-2172). N1. 2017. 385 pp.

Available at (Free):

http://www.math.spbu.ru/diffjournal/pdf/kuznetsov_book3.pdf

http://www.math.spbu.ru/diffjournal/pdf/kuznetsov_book2.pdf

Denotations

$$I_{l_1\ldots l_k T,t}^{(i_1\ldots i_k)}=\int\limits_t^T(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)},$$

$$I_{l_1\ldots l_k T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{W}_{t_1}^{(i_1)}\ldots d\mathbf{W}_{t_k}^{(i_k)},$$

$l_1,\ldots,l_k=0,\ 1,\ 2,\ldots$

For example:

$$I_{0_{T,t}}^{(i_1)}=\int\limits_t^T d\mathbf{W}_{t_1}^{(i_1)},$$

$$I_{1_{T,t}}^{(i_1)}=\int\limits_t^T(t-t_1)d\mathbf{W}_{t_1}^{(i_1)},$$

$$I_{00_{T,t}}^{*(i_1 i_2)}=\int\limits_t^{*T}\int\limits_t^{*t_2}d\mathbf{W}_{t_1}^{(i_1)}d\mathbf{W}_{t_2}^{(i_2)},$$

$$I_{10_{T,t}}^{*(i_1 i_2)}=\int\limits_t^{*T}\int\limits_t^{*t_2}(t-t_1)d\mathbf{W}_{t_1}^{(i_1)}d\mathbf{W}_{t_2}^{(i_2)},$$

$$I_{000_{T,t}}^{*(i_1 i_2 i_3)}=\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{W}_{t_1}^{(i_1)}d\mathbf{W}_{t_2}^{(i_2)}d\mathbf{W}_{t_3}^{(i_3)}.$$

Case of Legendre Polynomials

$$I_{0_{T,t}}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{2_{T,t}}^{(i_1)} = \frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right),$$

$$I_{00_{T,t}}^{*(i_1 i_2)q} =$$
$$= \frac{T-t}{2}\left[\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right\}\right],$$

$$I_{10_{T,t}}^{*(i_1 i_2)q} = -\frac{T-t}{2}I_{00_{T,t}}^{*(i_1 i_2)q} - \frac{(T-t)^2}{4}\left[\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)} + \right.$$
$$\left. + \sum_{i=0}^{q}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right],$$

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{*(i_1 i_2)} - I_{00_{T,t}}^{*(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right).$$

$$\mathsf{M}\left\{\left(I_{10_{T,t}}^{*(i_1 i_2)} - I_{10_{T,t}}^{*(i_1 i_2)q}\right)^2\right\} =$$
$$= \frac{(T-t)^4}{16}\left(\frac{5}{9} - 2\sum_{i=2}^{q}\frac{1}{4i^2-1} - \sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2} - \right.$$
$$\left. - \sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right).$$

Coefficients $\bar{C}_{0jk}$

$j \backslash k$	0	1	2	3	4	5	6
0	$\frac{4}{3}$	$-\frac{2}{3}$	$\frac{2}{15}$	0	0	0	0
1	0	$\frac{2}{15}$	$-\frac{2}{15}$	$\frac{4}{105}$	0	0	0
2	$-\frac{4}{15}$	$\frac{2}{15}$	$\frac{2}{105}$	$-\frac{2}{35}$	$\frac{2}{105}$	0	0
3	0	$-\frac{2}{35}$	$\frac{2}{35}$	$\frac{2}{315}$	$-\frac{2}{63}$	$\frac{8}{693}$	0
4	0	0	$-\frac{8}{315}$	$\frac{2}{63}$	$\frac{2}{693}$	$-\frac{2}{99}$	$\frac{10}{1287}$
5	0	0	0	$-\frac{10}{693}$	$\frac{2}{99}$	$\frac{2}{1287}$	$-\frac{2}{143}$
6	0	0	0	0	$-\frac{4}{429}$	$\frac{2}{143}$	$\frac{2}{2145}$

$$I_{000_{\tau_{p+1},\tau_p}}^{(i_1 i_2 i_3)q} = I_{000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)q} - \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^{3/2}\left(\zeta_0^{(i_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right) -$$

$$-\frac{1}{4}\mathbf{1}_{\{i_2=i_3\}}(T-t)^{3/2}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

$$I_{000_{\tau_{p+1},\tau_p}}^{*(i_1 i_2 i_3)q} = \sum_{j_1,j_2,j_3=0}^{q} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$

$$(i_1, i_2, i_3 = 1, \ldots, m)$$

$$C_{kji} = \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{8}(T-t)^{3/2}\bar{C}_{kji},$$

$$\bar{C}_{kji} = \int_{-1}^{1} P_k(z) \int_{-1}^{z} P_j(y) \int_{-1}^{y} P_i(x)dxdydz,$$

$P_i(x);\ i = 0,\ 1,\ 2, \ldots$ – *Legendre polynomials.*

Trigonometric Case (More Complex Formulas)

$$I_{0_{T,t}}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{2_{T,t}}^{(i_1)} = (T-t)^{5/2}\left[\frac{1}{3}\zeta_0^{(i_1)} + \frac{1}{\sqrt{2}\pi^2}\left(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)} + \sqrt{\beta_q}\mu_q^{(i_1)}\right) + \right.$$
$$\left. + \frac{1}{\sqrt{2}\pi}\left(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right],$$

$$I_{00_{T,t}}^{*(i_2 i_1)q} = \frac{1}{2}(T-t)\left[\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\left\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \right.\right.$$
$$\left.\left. + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right\} + \frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\right],$$

$$I_{10_{T,t}}^{*(i_2 i_1)q} = (T-t)^2\left(-\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)} - \frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_2)} + \right.$$
$$+ \frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(2\mu_q^{(i_2)}\zeta_0^{(i_1)} - \mu_q^{(i_1)}\zeta_0^{(i_2)}\right) +$$
$$+ \frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left[-\frac{1}{\pi r}\left\{\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \frac{1}{\sqrt{2}}\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \frac{1}{\sqrt{2}}\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\right\} + \right.$$
$$\left. + \frac{1}{\pi^2 r^2}\left(-\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)} + 2\zeta_0^{(i_1)}\zeta_{2r}^{(i_2)} - \frac{3}{2\sqrt{2}}\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \frac{1}{2\sqrt{2}}\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right)\right]$$
$$\left. + \frac{1}{2\pi^2}\sum_{\substack{k,l=1\\k\neq l}}^{q}\frac{1}{l^2-k^2}\left[\zeta_{2k}^{(i_1)}\zeta_{2l}^{(i_2)} - \frac{k}{l}\zeta_{2k-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\right]\right),$$

where

$$\xi_q^{(i)} = \frac{1}{\sqrt{\alpha_q}} \sum_{r=q+1}^{\infty} \frac{1}{r} \zeta_{2r-1}^{(i)},$$

$$\alpha_q = \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2},$$

$$\mu_q^{(i)} = \frac{1}{\sqrt{\beta_q}} \sum_{r=q+1}^{\infty} \frac{1}{r^2}\ \zeta_{2r}^{(i)},$$

$$\beta_q = \frac{\pi^4}{90} - \sum_{r=1}^{q} \frac{1}{r^4},$$

$$\zeta_0^{(i)},\ \zeta_{2r}^{(i)},\ \zeta_{2r-1}^{(i)},\ \xi_q^{(i)},\ \mu_q^{(i)} \sim N_{ind}(0,1)$$

$$r = 1, \ldots, q;\ i = 1, \ldots, m;\ i_1, i_2, i_3 = 1, \ldots, m.$$

5. COMPARISON WITH G.N.MILSTEIN METHOD OF STRONG APPROXIMATION OF MULTIPLE ITO AND STRATONOVICH STOCHASTIC INTEGRALS

[6] Milstein G.N. Numerical Integration of Stochastic Differential Equations. (In Russian). Sverdlovsk: Ural University Publ., 1988. 224 pp.

[7] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. *Stoch. Anal. Appl.* 10: 4 (1992), 431-441.

[8] Kloeden P.E., Platen E. Numerical solution of stochastic differential equations. Berlin, Springer-Verlag Publ., 1992. 632 pp.

The Idea of G.N.Milstein Method

Let's analyze the Brownian bridge process

$$\mathbf{W}_t^{(i)} - \frac{t}{\Delta}\mathbf{W}_\Delta^{(i)},\ t \in [0, \Delta],\ \Delta > 0;\ i = 1, \ldots, m. \qquad (3)$$

Let's consider the expansion of (3) into the trigonometric Fourier series converging in the mean-square sense

$$\mathbf{W}_t^{(i)} - \frac{t}{\Delta}\mathbf{W}_\Delta^{(i)} = \frac{1}{2}a_{i,0} + \sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \quad (4)$$

$$a_{i,r} = \frac{2}{\Delta}\int_0^\Delta \left(\mathbf{W}_s^{(i)} - \frac{s}{\Delta}\mathbf{W}_\Delta^{(i)}\right)\cos\frac{2\pi rs}{\Delta}ds,$$

$$b_{i,r} = \frac{2}{\Delta}\int_0^\Delta \left(\mathbf{W}_s^{(i)} - \frac{s}{\Delta}\mathbf{W}_\Delta^{(i)}\right)\sin\frac{2\pi rs}{\Delta}ds;\ r = 0,\ 1, \ldots$$

According to (4):

$$\mathbf{W}_t^{(i)} \approx \mathbf{W}_t^{(i)p} = \mathbf{W}_\Delta^{(i)}\frac{t}{\Delta} + \frac{1}{2}a_{i,0} + \sum_{r=1}^{p}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right). \quad (5)$$

Then [7, 8]:

$$\int_t^{*T} \ldots \int_t^{*t_2} d\mathbf{W}_{t_1}^{(i_1)} \ldots d\mathbf{W}_{t_k}^{(i_k)} \approx \int_t^{*T} \ldots \int_t^{*t_2} d\mathbf{W}_{t_1}^{(i_1)p_1} \ldots d\mathbf{W}_{t_k}^{(i_k)p_k}.$$

Disadvantages of G.N.Milsten method in comparison with the method, based on the theorem 1 [1-5] (D.F.Kuznetsov).

I. There is no obvious formula for calculation of expansion coefficients of multiple stochastic integrals.

II. Practically impossible to calculate exactly the mean-square errors of approximations of multiple stochastic integrals. It is possible only in simplest cases.

III. The basis functions is only trigonometric functions.

IV. G.N.Milstein method leads to repeated series (in contrast with multiple series taken from theorem 1 [1-5]) starting at least from the 3rd multiplicity of multiple stochastic integral:

$$\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} \quad \textbf{in the theorem 1}$$

$$\lim_{p_1\to\infty}\sum_{j_1=0}^{p_1}\ldots\lim_{p_k\to\infty}\sum_{j_k=0}^{p_k} \quad \textbf{in G.N.Milstein method}$$

So, expansions from the theorem 1 [1-5] converges under condition

$$p_1=\ldots=p_k=p\to\infty, \tag{6}$$

but expansions from G.N.Milstein method may not converges under condition (6).