A Theorem on the Expansion of Multiple Stochastic Stratonovich Integrals of Any Arbitrary Multiplicity k, Based on the Repeated Fourier Series

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Let $J^*[\psi^{(k)}]_{T,t}$ be a multiple Stratonovich stochastic integral:

$$J^*[\psi^{(k)}]_{T,t} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a smooth function on $[t, T]$; $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$; $\mathbf{f}_\tau$ is a standard m-dimensional Wiener stochastic process with independent components $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$; $i_1, \ldots, i_k = 0,\ 1, \ldots, m$.

Define the following function on a hypercube $[t, T]^k$:

$$K^*(t_1, \ldots, t_k) = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} \left(\mathbf{1}_{\{t_l < t_{l+1}\}} + \frac{1}{2} \mathbf{1}_{\{t_l = t_{l+1}\}} \right);\ t_1, \ldots, t_k \in [t, T];\ k \geq 2,$$

and

$$K^*(t_1) = \psi_1(t_1);\ t_1 \in [t, T],$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials or trigonometric functions in $L_2([t, T])$.

It was shown in [1], [3] - [8] that

$$K^*(t_1, \ldots, t_k) = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l),\ (t_1, \ldots, t_k) \in (t, T)^k, \tag{1}$$

where

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K^*(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k, \tag{2}$$

and the repeated Fourier series (1) converges at the boundary of hypercube $[t, T]^k$.

Theorem 1 (see [1] – [8]). *Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuously differentiable on $[t, T]$ function and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in $L_2([t, T])$. Then, the multiple Stratonovich* *stochastic integral* $J^*[\psi^{(k)}]_{T,t}$ *can be expressed as the converging in the mean of degree* $2n$ $(n \in N)$ *repeated series*

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}, \tag{3}$$

where every

$$\zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

is a standard Gaussian random variable for various i *or* j *(if* $i \neq 0$*) and* $C_{j_k \ldots j_1}$ *is the Fourier coefficient* (2).

The relation (3) means the following

$$\lim_{p_1 \to \infty} \ldots \lim_{p_k \to \infty} \mathsf{M}\left\{\left(J^*[\psi^{(k)}]_{T,t} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}\right)^{2n}\right\} = 0, \ n \in N.$$

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