A Theorem on the Expansion of Multiple Stochastic Ito Integrals of Any Arbitrary Multiplicity k, Based on the Multiple Fourier Series Converging in the Mean-Square Sense

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Let $J[\psi^{(k)}]_{T,t}$ be a multiple Ito stochastic integral:

$$J[\psi^{(k)}]_{T,t} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous function on $[t, T]$; $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$; $\mathbf{f}_\tau$ is a standard m-dimensional Wiener stochastic process with independent components $\mathbf{f}_\tau^{(i)}$ $(i = 1, \ldots, m)$; $i_1, \ldots, i_k = 0,\ 1, \ldots, m.$

Define the following function on a hypercube $[t, T]^k$:

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k),\ t_1 < \ldots < t_k \\ \\ 0,\ \text{иначе} \end{cases};\ t_1, \ldots, t_k \in [t, T];\ k \geq 2,$$

and

$$K(t_1) = \psi_1(t_1);\ t_1 \in [t, T].$$

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in $L_2([t, T])$. The function $K(t_1, \ldots, t_k)$ is sectionally continuous in the hypercube $[t, T]^k$. At this situation it is well known, that the multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the hypercube $[t, T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1, \ldots, p_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right\| = 0,$$

where

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k, \qquad (1)$$

$$\|f\|^2 = \int\limits_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t, T]$ such that

$$t = \tau_0 < \ldots < \tau_N = T,\ \Delta_N = \max_{0 \leq j \leq N-1} \Delta\tau_j \to 0 \text{ if } N \to \infty,\ \Delta\tau_j = \tau_{j+1} - \tau_j. \qquad (2)$$

Theorem 1 (see [1] - [11]). *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous on* $[t,T]$ *function and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of continuous functions in* $L_2([t,T])$. *Then*

$$J[\psi^{(k)}]_{T,t}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathcal{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr),$$

where

$$\mathcal{G}_k=\mathcal{H}_k\backslash\mathcal{L}_k;\ \mathcal{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\};$$

$$\mathcal{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\};$$

l.i.m. *is a limit in the mean-square sense;* $i_1,\ldots,i_k=0,1,\ldots,m$; *every*

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

is a standard Gaussian random variable for various i *or* j (*if* $i\ne 0$); $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (1); $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$; $\{\tau_j\}_{j_l=0}^{N-1}$ *is a partition of* $[t,T]$, *which satisfies the condition* (2).

It was shown in [3] - [11] that theorem 1 is valid for convergence in the mean of degree $2n$, $n\in N$.

In order to evaluate significance of the theorem 2 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,4$:

$$J[\psi^{(1)}]_{T,t}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)},$$

$$J[\psi^{(2)}]_{T,t}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Bigl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Bigr),$$

$$J[\psi^{(3)}]_{T,t}=\underset{p_1,\ldots,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3j_2j_1}\Bigl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Bigr),$$

$$J[\psi^{(4)}]_{T,t}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr),$$

where $\mathbf{1}_A$ is the indicator of the set A.

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